SECOND COURSE IN ALGEBRA

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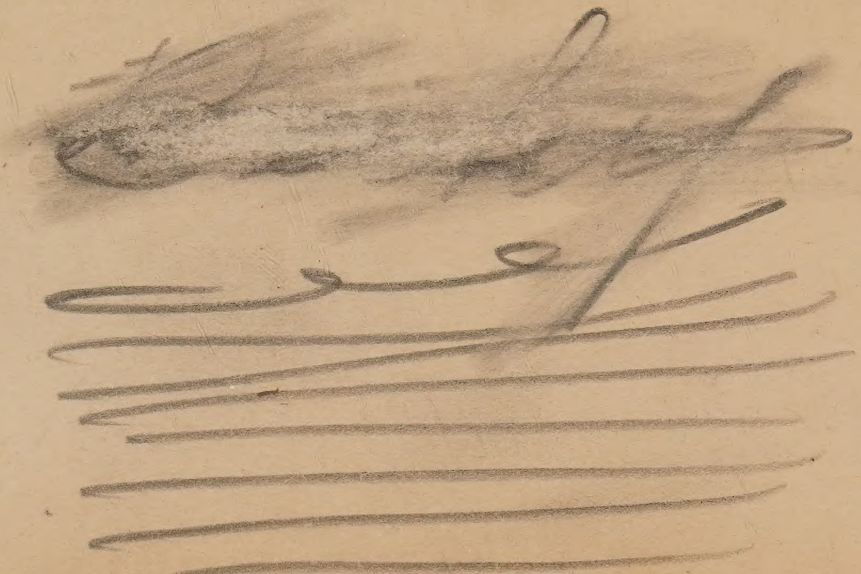

Gottfried Wilhelm von Leibnitz

A SECOND COURSE

IN

ALGEBRA

BY

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WEST HIGH SCHOOL, ROCHESTER, NEW YORK

ALLYN AND BACON

BOSTON NEW YORK CHICAGO
ATLANTA SAN FRANCISCO

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PREFACE

This book is intended primarily to follow our *First Course in Algebra,* but it is well adapted for use after any good first-year text. It gives a brief but thorough review of the topics of the First Course, and includes a sufficient amount of advanced material to meet the needs of any class in second-year or intermediate algebra, whether the course be a half year or a year in length. The content of the book fully meets the requirements for college entrance.

The leading features of our First Course have been incorporated in this book. Among them we call attention to the simplicity of the treatment, the introductory paragraphs which link each chapter to that which has gone before, the number and arrangement of the illustrative examples, the great number of word problems, the extent and variety of the graphic material, and the emphasis which has been laid upon the formula.

As in our First Course the pupil is led to review constantly the work he has had by means of short reviews at the end of most of the exercises. Besides these an extended general review is provided at the end of the book for classes which need it.

We have tried in this book to develop the function concept which is so necessary for success in higher mathematics. In developing this and other topics, however, we have attempted to maintain the same simplicity which is a feature of our First Course. We have not lost sight of the fact that we are writing for boys and girls of immature minds, and not for experienced mathematicians.

The problem material is abundant and varied. Simple applications of geometry, such as the pupils have already had in their arithmetic, have been freely used. The field of business also furnishes many problems. We believe that the wide variety of topics will provide something to stimulate the interest of all.

The syllabus of the College Entrance Board and the recommendations of the National Committee on the Reorganization of Mathematics in Secondary Education have been constantly kept in mind, and we have also tried as far as possible to meet the local requirements of the various states.

We gratefully acknowledge the helpful suggestions which we have received from those who are using our First Course, and in particular from our colleagues, Mr. Nelson L. Roray of the William L. Dickinson High School, Jersey City, and Mr. L. A. Pultz of the West High School, Rochester.

E. I. Edgerton
P. A. Carpenter

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SECOND COURSE IN ALGEBRA

CHAPTER I

FUNDAMENTAL OPERATIONS

In the study of algebra the pupil should not forget that the quantities with which he is dealing are the same as those of arithmetic. That is, they are numbers rather than the space magnitudes of geometry. The quantities of arithmetic are *specific* numbers, while those of algebra are *general* numbers. This is the chief difference between these subjects.

1. Evaluation. — The fact that the letters of algebra represent numbers is made clear by the many instances where it becomes necessary to find the numerical value of an algebraic expression when specific values are assigned to the letters involved.

Illustrative example I:

Find the value of

$$\frac{a}{b}+\frac{b-d}{ac} \text{ if } a=4,\ b=3,\ d=1, \text{ and } c=2.$$

Solution: $\frac{4}{3}+\frac{3-1}{4\cdot 2}=\frac{4}{3}+\frac{2}{8}=\frac{4}{3}+\frac{1}{4}=\frac{19}{12}.$

Illustrative example II:

Evaluate $\frac{ar^n-a}{r-1}$ if $a=3$, $r=2$, and $n=6$.

Solution:

$$\frac{3\cdot 2^6-3}{2-1}=\frac{3\cdot 64-3}{1}=192-3=189.$$

Exercises

1. Does $4 \cdot 3 - 1$ equal 11 or 8?
2. Does $12 - 6 \div 2 = 3$ or 9?
3. Does $40 \div 2 \cdot 5 = 4$ or 100?
4. Does $2 \cdot 3^2 = 18$ or 36?
5. Explain difference of meaning in each of the following:

(a) $ab + c$ and $a(b + c)$.

(b) $a - 3x$ and $(a - 3)x$.

(c) ax^2 and $(ax)^2$.

(d) $a - \frac{b}{c}$ and $\frac{a - b}{c}$.

Find the value of each of the following when $a = 3$, $b = 1$, $c = 5$, $d = 0$, and $e = 4$.

6. $3a^2 + 4ab + b^2$.
7. $(a - 2)(b + 3)$.
8. $\frac{2a + e}{c} + \frac{d}{a}$.
9. $ab + bc + cd + de$.
10. $(a + b + c + e)d$.
11. $a^b + c^e$.
12. $\frac{ab}{c^2} + \frac{bd}{e^2} + \frac{ae}{b^2}$.

Evaluate

13. $\frac{(b + b')h}{2}$ if $b = 15$, $b' = 8$, and $h = 12$.
14. $\pi(R^2 - r^2)$ if $R = 14$, $r = 6$.
15. $\pi r^2 h$ if $r = 2.3$ and $h = 4.8$.
16. $\frac{4}{3}\pi r^3$ if $r = 2.1$.
17. $\frac{n}{2}(a + l)$ if $n = 10$, $a = 6$, and $l = 24$.
18. $\sqrt{s(s - a)(s - b)(s - c)}$ if $a = 6$, $b = 4$, $c = 8$, and $s = \frac{1}{2}(a + b + c)$.
19. Can we evaluate $\frac{ab^2}{c}$ if $a = 2$, $b = 3$, $c = 0$? Why?
20. Find the value of $\frac{a^3 - 3a^2 + 3a - 1}{a + 3}$ when $a = 3$.

21. What is the value of $\frac{-b-\sqrt{b^2-4ac}}{2a}$ if $a=2, b=5$, and $c=2$?

22. What is the value of

$$a-(a-b)+[2a-(b+2c)]-[2a-b-(c-d)]$$

if $a=6, b=2, c=0, d=2$?

23. Will $x=20$, or $x=40$, or $x=10$ check in the equation

$$2+\frac{3x}{4}-\frac{4x}{5}=0?$$

24. What is the value of $(a+b)(a-b)$ if $a=.2$ and $b=.03$?

25. Find the value of i in $i=prt$, when the principal $(p)=\$600$, the rate $(r)=6\%$, and the time $(t)=4$ years.

26. Using the formula of example 25, find the interest (i) if $p=\$400$, $r=5\%$, and $t=3$ years 6 months. If $p=\$250$, $r=4\frac{1}{2}\%$, and $t=3$ years 3 months.

27. Given that the value of x is 5, find the value of each of the following: $.5x+.08x^3$, $.4x-2$, $\frac{3}{4}x+.8$, $.56x^2$, $x+2xx-2$.

28. If $y=$ yards and f represents feet, what is the value of $8y+6f$ in inches?

29. If $f=12i$, how many i's in $5f+8f+10f$?

30. If you are $(13y+7m+12d)$ old, how many days is this?

31. If $1q=25c$, how many c's in $7q+6q+2q$?

32. Find p (perimeter of a rectangle) if $p=2(l+w)$, when $l=.75$ of a foot, and $w=.15$ of a foot.

33. If $A=lw$, what is the value of A when (a) $l=3.5$, $w=.25$; (b) $l=.25$, $w=.47$; (c) $l=10$, $w=3.45$; (d) $l=1000$, $w=34.6$?

Exercises

Using proper algebraic symbols, write the following:

1. 6 minus y plus b.

2. 6 times 3 x as the sum of six terms. Find the sum.

3. The perimeter of a square, 48 x, divided by 4. What does your result represent?

4. John has n nickels and his brother has 5 times as many nickels and 3 nickels more.

5. The number of girls in a certain class is three less than twice the number of boys (b). How many girls are there?

6. One number is twice a second number (n). A third number is 4 less than the first number. A fourth number is four times the second number. What represents the sum of all the numbers?

7. There are three consecutive odd whole numbers. If n represents the middle number, what represents the sum of all the numbers?

8. If Henry sold six times as many papers today as his brother, and n represents the number sold by the latter, what expression represents the difference of the numbers sold by each?

2. Processes with Positive and Negative Numbers. — Ability to perform all fundamental operations on positive and negative numbers is of prime importance. You should memorize the following rules governing their use.

Rules. — (1) *Addition: To add two numbers having like signs, find the sum of their absolute values and prefix the common sign.*

To add two numbers having unlike signs, find the difference of their absolute values and prefix the sign of the greater.

(2) *Subtraction: Mentally change the sign of the subtrahend and proceed as in algebraic addition.*

(3) *Multiplication: The product of two numbers having like signs is positive.*

The product of two numbers having unlike signs is negative.

(4) *Division: The quotient of two numbers having like signs is positive.*

The quotient of two numbers having unlike signs is negative.

Exercises

Add:

1.	2.	3.	4.	5.	6.
-6	-7	$+6$	-8	-3	-5
$+3$	$+4$	$+2$	$+7$	-9	-10
$+10$	-3	-7	$+2$	-10	-3
-2	-8	-10	-1	$+4$	-8
$+6$	-2	-3	$+6$	-3	-4

7. Combine: $6+(-3)+(-5)+(+12)+6+(-3)+5$.

8. Subtract:

10	8	4	6	0	-3	-3	-5	$+6$
-2	-5	-10	0	5	6	-1	$+4$	$+2$

9. Multiply:

-6	-10	-4	$+3$	-10	-20	-6	-8	-10
$+2$	-3	$+6$	$+2$	5	$+3$	-2	-3	$-\frac{1}{2}$

10. Divide:

$\frac{10}{-2}$,	$\frac{-8}{-4}$,	$\frac{-12}{+3}$,	$\frac{+6}{+2}$,	$\frac{-14}{-7}$,	$\frac{0}{-6}$,	$\frac{+\frac{1}{2}}{-2}$,	$\frac{-6}{-\frac{1}{2}}$

Find the value of

11. $2-(-6)+(-3)-2(-2)$.

12. $6(-3)-4(-6)+(3)^2$.

13. $(-1)^2+(-2)^3+(-1)(-2)(-3)$.

14. $\frac{6}{-2}+\frac{-10}{5}-(-3)+1^2$.

15. $6(-2)+\frac{-10}{-5}-(-3)^2+0\cdot 1(-6)$.

16. Evaluate $a^2+ab+ac-b^2-c^2$ if $a=2$, $b=-2$, and $c=-3$.

17. Evaluate $a^x+2b^y-3c^z+5w^a$ if $a=2$, $b=1$, $c=-1$, $x=3$, $y=4$, $z=6$, $w=-3$.

3. Definitions. — It is of course assumed that you are familiar with the meaning of the more common algebraic words. A few of them are of special importance at this time.

A *term* is that part of an algebraic expression included between two adjacent + or − signs. A fraction, or a quantity inclosed within a parenthesis, is considered as a single quantity, and hence a term or part of a term.

A *monomial* is an expression of one term, as $3x^2$, $-6xy(a+b)$, $\frac{-3ab}{2}$.

A *binomial* is an expression of two terms, as $a+b$, $3x+2mn^2$, $(a+b)^2-(c+d)^2$.

A *trinomial* is an expression of three terms. State one.

A *polynomial* is an expression of two or more terms. Mention one.

Similar terms are terms which are alike in the literal parts. They may differ only in signs and coefficients. Examples: $3ax^2$, $-6ax^2$, $10ax^2$; $5(a+b)$, $3(a+b)$, $-10(a+b)$.

4. Addition of Monomials and Polynomials. — Only similar terms may be added. The addition of dissimilar terms may, however, be indicated. Thus the sum of a and b is $a+b$.

Rule. — *To add two or more similar terms, combine their coefficients by the rule of section 2, and annex the literal parts.*

The work may be arranged horizontally or vertically.

Illustrative example I:

$$\begin{array}{r} -3x^2y \\ -6x^2y \\ -2x^2y \\ +7x^2y \\ \hline -4x^2y \end{array}$$

Illustrative example II:

$3a(b+c) - 6a(b+c) - 2a(b+c) + 12a(b+c) = 7a(b+c)$.

Rule. — *To add two or more polynomials, arrange them so that similar terms fall in vertical columns and add each column separately.*

Illustrative example:

Add: $3x^3 - 4x^2 + 6x - 2$; $x^2 - 9x + 6$; $-3x^3 + 4$; $-2x^2 - 8x + 3$; $x^3 - 2x^2 - 9x + 6$; $x^3 - x + 4$.

Solution:

				Check when $x = 2$.
$3x^3$	$-4x^2$	$+6x$	-2	$= 18$
	x^2	$-9x$	$+6$	$= -8$
$-3x^3$			$+4$	$= -20$
	$-2x^2$	$-8x$	$+3$	$= -21$
x^3	$-2x^2$	$-9x$	$+6$	$= -12$
x^3		$-x$	$+4$	$= 10$
$2x^3$	$-7x^2$	$-21x$	$+21$	$= -33$

5. Check of Addition. — Addition may be checked by assuming a numerical value other than 1 or 0 for each letter concerned. A different value should be used for each different letter. The above example was checked by assuming that $x = 2$.

Subtraction is also a check of addition, and is especially useful if only two expressions are added. When one of these expressions is subtracted from the sum, the result should equal the other expression.

Exercises

Add each of the following:

1. $3x^2y$, $-4x^2y$, $10x^2y$, $2x^2y$, $-6x^2y$

2. $4a(m+n)$, $6a(m+n)$, $-3a(m+n)$, $-2a(m+n)$, $a(m+n)$

3. $-5x^2y^3$, $-8x^2y^3$, $-10x^2y^3$, $-x^2y^3$, $-6x^2y^3$

4. $4\frac{xy}{z}$, $10\frac{xy}{z}$, $-3\frac{xy}{z}$, $-15\frac{xy}{z}$

5. Collect: $3a^2b + 5a^2b - 10a^2b + 2a^2b - a^2b - 8a^2b + 6a^2b$.

6. Add: $a^2b + 3ab + 6ab^2$; $2a^2b - 5ab - 6ab^2$; $3a^2b + 10ab - 4ab^2$; $3a^2b - 2ab + 10ab^2$.

Check by letting $a = 2$ and $b = 3$.

7. Add: $2a^3 - 6a^2 + 2a + 3$; $7a^3 + 4a^2 - a$; $a^2 + 6a - 9$; $7a^3 + a - 2$; $a^2 + 2a - 6$.

Check with $a = 2$.

8. Add:

$$\begin{array}{r} 2(a+b) - 3(c+d) - 6(m+n) \\ 4(a+b) + 4(c+d) - 3(m+n) \\ -10(a+b) - 2(c+d) + 15(m+n) \\ (a+b) - (c+d) - (m+n) \\ \underline{2(a+b) \qquad\qquad + 6(m+n)} \end{array}$$

9. Add:

$$\begin{array}{r} 2\,a^3 + 3\,a^2b - 4\,ab^2 + b^3 \\ a^3 - 10\,a^2b - 2\,ab^2 - b^3 \\ 3\,a^3 + 2\,a^2b + 6\,ab^2 - 5\,b^3 \\ -6\,a^3 + 5\,a^2b + 5\,b^3 \\ \hline 2\,a^3 + 6\,a^2b - 8\,ab^2 - b^3 \end{array}$$

How does the answer to this example compare with the last expression which was added? Why?

10. Combine:

$ax^2 - 6\,a^2x + 3\,ax - 4\,a^2x - 6\,ax^2 - 2\,a^2x + 5\,ax - 3\,ax^2 - 5\,ax - 10\,a^2x + 4\,ax^2 - ax + 3\,a^2x - 3\,ax^2 + 2\,ax - 2\,a^2x + 10\,ax^2.$

11. Add:

$2.5\,x$	$.25\,x$	$(a+b)x$	$.246\,x$
$-1.6\,x$	$.56\,x$	$2(a+b)x$	$-.345\,x$
$3.7\,x$	$-.65\,x$	$6(a+b)x$	$-.125\,x$
$-4.8\,x$	$.15\,x$	$-4(a+b)x$	$.375\,x$

Add and check:

12. x^2yz^3, $-2\,x^2yz^3$, $+6\,x^2yz^3$, $-3\,x^2yz^3$.

13. $.6\,x$, $.4\,x$, $.5\,x$, $-.3\,x$, $+.7\,x$, $-.9\,x$, $-.2\,x$.

14. $\frac{1}{2}\,xy$, $\frac{1}{3}\,xy$, $\frac{1}{4}\,xy$, $\frac{1}{6}\,xy$. Check with $x = 4$ and $y = 2$.

15. $6\,x^2y$, $4\,x^2y$, $8\,x^2y$, $-5\,x^2y$, $\frac{2}{3}\,x^2y$, $3\,x^2y$.

16. x^a, $2\,x^a$, $-5\,x^a$, $-6\,x^a$, $3\,x^a$.

17. x^{a+1}, $3\,x^{a+1}$, $5\,x^{a+1}$, $-6\,x^{a+1}$, $-2\,x^{a+1}$. Check with $x = 2$ and $a = 3$.

18. $.6\,x^2y$, $\frac{1}{2}\,x^2y$, $.36\,x^2y$, $\frac{1}{4}\,x^2y$.

19. $3\,x^3y^{a+2}$, $-5\,x^3y^{a+2}$, $-7\,x^3y^{a+2}$, $+4\,x^3y^{a+2}$. Check with $x = 2$, $y = 3$, and $a = 4$.

6. Subtraction of Monomials and Polynomials. — Subtraction of monomials and polynomials may be performed by the following

Rule. — (1) *Set the subtrahend under the minuend so that similar terms fall in vertical columns.*

(2) *Subtract the numerical coefficients by rule of section 2 and bring down the literal parts.*

(3) *If the coefficients are literal, their difference will have to be indicated.*

The subtraction of dissimilar terms can only be indicated. For example if we subtract $3\,y$ from $7\,x$, we indicate the result as $7\,x - 3\,y$.

Illustrative example:

From $3\,x^3 - 6\,x^2 + x - 2$ subtract $x^4 - 5\,x^2 - 4\,x + 1$.

Solution:

$$\begin{array}{rl} & 3\,x^3 - 6\,x^2 + x - 2 \quad \text{minuend} \\ x^4 & - 5\,x^2 - 4\,x + 1 \quad \text{subtrahend} \\ \hline -x^4 + 3\,x^3 - & x^2 + 5\,x - 3 \quad \text{remainder.} \end{array}$$

7. Checks of Subtraction. — Subtraction may be checked, as in addition, by substituting a suitable numerical value for each of the literal quantities involved. For example, in the above problem we may substitute 2 for x. Then the minuend equals 0, the subtrahend -11, and the remainder 11. Since $0 - (-11) = 11$ the work is correct.

Subtraction may also be checked by addition. The minuend is the sum of the subtrahend and remainder.

Exercises

Perform each of the indicated subtractions and check your work:

1.	2.	3.	4.
$6\,a^2y^3$	$-10\,aym$	$4\,x^3yz^2$	$-11\,yzx^2$
$-4\,a^2y^3$	$-2\,aym$	$10\,x^3yz^2$	$-15\,yzx^2$

5. $2a - 6b$ / $4a - 3b$

6. $7x - 2y^2$ / $12x + 5y^2$

7. $-8x + 3y - z$ / $2x - 6y + 4z$

8. $3x^3 - 6x^2 + 4x - 10$ / $2x^3 + 5x^2 - 6x + 2$

9. $-6a^2 - 7a + 3$ / $12a^2 + 7a + 3$

10. From the sum of $2x^2 - 8x + 6$ and $x^3 - x^2 + 9x - 3$ subtract the sum of $2x^3 + 5x - 9$ and $x^2 + 8x + 1$.

11. What must be added to $6x^2 - 8xy + 4y^2$ to produce $3x^2 - 2xy + 7y^2$?

12. What must be added to $8a^3 - 7a^2b - 6ab^2 - 10$ to produce 0?

13. By how much does $-4a^2b - 3ab^2 + b^3$ exceed $a^3 + 2a^2b + 6ab^2 - b^3$?

Subtract the first expression from the second:

14. $6x$, $5x$.

15. $7xy$, $-3xy$.

16. $9x^2$, $5x^2$.

17. $3xyz$, $-6xyz$.

18. $2x + y$, $3x - 2y$.

19. $5x^2 - y^2$, $4x^2 + y^2$.

20. $7xy + 2zw$, $5xy - 4zw$.

21. $5x + 7y - 3z$, $3x + 4y - 2z$.

22. $5x^3 - 6x$, $4x^2 + 7x$.

23. $8x^2 + 5x$, $6x^3 - 7x$.

24. $2xy + 6yz - 4zw$, $5xy - 6yz + 4zw$.

Subtract:

25. $.5x^2yz$ from $.2x^2yz$.

26. $\frac{1}{4}xy^2z^3$ from $\frac{1}{2}xy^2z^3$.

27. $3.6x^{a+2}$ from $7.8x^{a+2}$. Check with $x = 3$ and $a = 2$.

28. $9.6x^3$ from $10\frac{1}{2}x^3$.

29. $5.7x^3 - 8.6x^3 = ?$

30. $\frac{7}{8}x^5 - \frac{1}{4}x^5 = ?$

31. $12x^6 - ? = 7x^6$.

32. $8.5x^2 - ? = 5.2x^2$.

33. $6(x+y)^3 - 3(x+y)^3 = ?$

34. $5(x + y) - ? = 2(x + y)$.

35. $3.5(x + y)^2 - ? = 1.6(x + y)^2$.

Subtract the second expression from the first:

36. $5x, 6x$.

37. $7.6x, 3.4x$.

38. $2.8x^2, 5.7x^2$.

39. $6(x+y), 8(x+y)$.

40. $7x^{a+3}, 9x^{a+3}$. Check with $x = 3$ and $a = 2$.

41. $5(x+y)^a - 4.6(x+y)^a - 8.7(x+y)^a = ?$

42. $6.8(x+y)^a - 5.4(x+y)^a - 3.2(x+y)^a = ?$

Review

1. Add: $x^2 + 5xy + h, 4x^2 - 6xy + 4h, x^2 - 2xy - 3h$.

2. From the sum of $5x - 8y + 3z$ and $6x - 4z + 6y$ subtract the sum of $7z + 5y - 2x$ and $3y - 5z + 9x$.

3. Solve and check:

$$4x + .24 = .6x - 12.$$

What is the root of an equation? How many roots should you expect from the above equation? Why?

4. Is 3-tenths a root of the equation $\frac{1}{2}x + \frac{1}{3}x = \frac{1}{4}$? Prove.

5. Write with algebraic symbols:

(a) x minus 5 times y.

(b) x divided by y and the result added to z.

(c) x divided by the sum of y and z.

(d) The product of x and y divided by z.

(e) The cube of x multiplied by the square of y and the result subtracted from the fourth power of z.

6. Translate each of the following algebraic statements into the equivalent word statement:

(a) $x + 5 = 15$.

(b) $2x + 6 = 18$.

(c) $3x - 7 = 28$.

(d) $.5x - 9 = 16$.

(e) $\frac{x}{4} + 2 = 10$.

(f) $6x = 18$.

(g) $\frac{x}{2} - \frac{x}{3} = 30$.

(h) $5\left(\frac{x}{2} + \frac{x}{3}\right) = 10$.

7. If $x = 2y$, what is done to give $4x = 8y$? $\frac{1}{2}x = y$? $x + 3 = 2y + 3$? $x - 2 = 2y - 2$?

8. What exponent and what coefficient is x understood to have?

9. State two formulas and explain what each means. Show how you would evaluate each of them.

10. If $9x + 3 = x + 21$, why does $8x + 3 = 21$?

11. If $8x + 5 = x + 26$, why does $7x = 21$?

12. A man is now y years old. How old was he 8 years ago? x years ago? How old will he be 6 years hence? x years hence?

13. A person has d dollars. How much has he after he has earned 5 dollars? After he has lost x dollars? After he has trebled the original amount and then lost y dollars?

8. The Parenthesis. — The parenthesis in its various forms is used primarily to show that the quantities inclosed within it are to be treated as a single quantity. It furnishes a convenient way of indicating subtraction as

$$(6x^3 - 8x^2 - x + 3) - (2x^3 + 7x^2 + x - 1).$$

This means that the latter quantity is to be subtracted from the former. Since we change the signs of the subtrahend in subtraction and add, the above expression is equivalent to

$$6x^3 - 8x^2 - x + 3 - 2x^3 - 7x^2 - x + 1$$

or $$4x^3 - 15x^2 - 2x + 4.$$

Hence we have the following

Rule. — *A parenthesis preceded by a + sign expressed or implied may be removed without any change of signs; a parenthesis preceded by a − sign may be removed provided the sign of every term inclosed within it be changed.*

When a series of parentheses occurs, one within another, it is usually best to remove them one at a time, beginning with the innermost.

Illustrative example:

Simplify:

$2x - [3x + 6 + \{-2x - (3x - 1) + (x - 5)\} - 6x] + 3.$

Solution: Removing in order the parentheses, braces, and brackets, we obtain in succession

$$\begin{aligned} &2x - [3x + 6 + \{-2x - 3x + 1 + x - 5\} - 6x] + 3 \\ &= 2x - [3x + 6 - 2x - 3x + 1 + x - 5 - 6x] + 3 \\ &= 2x - 3x - 6 + 2x + 3x - 1 - x + 5 + 6x + 3 \\ &= 9x + 1. \quad \textit{Ans.} \end{aligned}$$

Evidently a similar rule must hold for the insertion of parentheses. Any desired terms may be inclosed by a parenthesis preceded by a + sign without change, but if terms are inclosed within a parenthesis preceded by a − sign, the signs of every term so inclosed must be changed.

Illustrative example:

In the expression $2x^2 + b - 3y - a + 2c - 3z$, let it be required to inclose by a parenthesis preceded by a + sign every term containing a, b, or c; and to inclose within a parenthesis preceded by a − sign every term containing x, y, or z. We have $+(b - a + 2c) - (-2x^2 + 3y + 3z)$.

Exercises

Remove parentheses and simplify:

1. $2 - [1 + 2 - (3 - 1) + 2] - 3.$
2. $5 + \{-6 - (-5 + 3) + (-6 + 2) - 3\} + 10.$
3. $-3a - \{2x + [2a - 3x + (a - 5)] - 3x + 6\}.$
4. $5a^2 - [6a^2 + 2a - 3a^2 - 1 + 10a] - 5.$

5. $(x - a - 1) - (a + x + 1) - (-2a - 2)$.

6. $3x^2 + [6x^2 - \{3x^2 + (2x^2 - 1) + 4\} - 3] + 5$.

7. $-\left(\frac{3}{4} + \frac{5a}{6} - \overline{2 - 8a}\right)$
$-\{-3 - [-2a - (6 + 5a - 2 + 3a)]\}$.

8. Find value of
$a - \{5b - [a - (3c - 3b) + 2c - 3(a - 2b - c)]\}$
if $a = -3$, $b = 4$, $c = -5$.

Simplify:

9. $a + b - \{-[a + b - (c + x)] - [3a - (c - x + a) - b] + 4a\}$.

10. $1 - \{1 - [1 - (1 - x)]\}$.

11. $(1 - 2x + 3x^2) + (3 + 2x - x^2)$.

12. $(a - 2b - 3c) - (b + c - 3d) + (2e + d - f)$
$-(f + g - e)$.

13. $x - [x - \{x - (x - \overline{x + x})\}]$.

Inclose terms containing a, b, and c within a parenthesis preceded by a + sign, and terms containing x, y, and z within a parenthesis preceded by a − sign.

14. $6a + 5x - 3z + y - b + 4c$.

15. $3y - 6b + 11c - x + 3z - a$.

16. $5x^2 - 11y^2 - 6a^2 + b^2 - c^2 + 3z^2$.

17. $5a^2 - 6bc - 8x^2 + 9y^3 - 7z^4$.

18. Simplify $.5x + (.65x - .4) - [.4x - (.3x - .7)]$.

19. Indicate the fact that the difference between a and b is to be added to 3 and this result is to be subtracted from $2a$.

20. Express the fact that the sum of x and y is to be subtracted from $3x$ and that the new remainder is to be subtracted from 50.

21. Indicate the subtraction of the sum of $3x$ and $4y$ from x.

22. What indicates the size of the third angle of a triangle if the other angles are $x°$ and $y°$?

23. How many degrees are there in an angle if its complement is $(5x + 40)$ degrees? If its supplement is $(8x + 15)$ degrees? What represents the difference between its supplement and its complement?

24. If the two known angles of a triangle are respectively $(x + 3y + 5)$ and $(3x - 4y - 20)$ degrees, what represents the third angle?

25. Indicate what must be added to $4x - 7$ to produce $5 - 2x$.

26. Indicate the result of subtracting the difference of the two following polynomials from their sum, $5x + 7y - 8z$, $8x + 3y + 5z$.

Review

Write in the simplest form:

1. $2x^2 + 3 - 5x^2 - 7 + 4x^2 - 6$.

2. $-2b + c + 5b - 4c - 2c - 6b$.

3. $8a^3 - 2b + 7a^3 + 3b - 15a^3 - b$.

4. $5 - 3x^2 + 4 - x^2 + 2x^2 - 8$.

5. $-ax - 5ax + b - 2b + 4ax + 5b$.

9. Exponents in Multiplication. —

Since $x^2 = x \cdot x$

and $x^3 = x \cdot x \cdot x$,

it follows that $x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x$

$= x^5$.

This illustrates the law of exponents in multiplication which may be expressed in symbols as follows:

$$x^m \cdot x^n = x^{m+n}.$$

This principle may be stated in words as follows:

The exponent of a given letter in a product is equal to the sum of the exponents of that letter in the multiplicand and the multiplier.

10. Multiplication by a Monomial. — To multiply a polynomial by a monomial, multiply each term of the multiplicand in succession by the multiplier, observing the laws of signs and exponents already stated, and write the resulting terms in order.

Illustrative examples:

(1) Multiply $-6\,a^3b^2$ by $3\,ab^3c$.

(2) Multiply $3\,a^2 - 6\,ax + 2\,x^2 + 1$ by $-2\,ax^4$.

Solutions:

(1)
$$\begin{array}{r} -6\,a^3b^2 \\ 3\,ab^3c \\ \hline -18\,a^4b^5c \end{array}$$

(2)
$$\begin{array}{r} 3\,a^2 - 6\,ax + 2\,x^2 + 1 \\ -2\,ax^4 \\ \hline -6\,a^3x^4 + 12\,a^2x^5 - 4\,ax^6 - 2\,ax^4 \end{array}$$

Oral Exercises

Multiply:

1. $a^2 \cdot a^3 \cdot a^8$.
2. $(-a^3)(-a^5)(a^8)$.
3. $-x^2(-x^5)$.
4. $x^2 \cdot x^3 \cdot x^4 \cdot x^{10}$.
5. $x^n \cdot x^n$.
6. $x^n(x^{n+1})$.
7. $(x^{n+1})^2$.
8. $x^{n+1} \cdot 2\,x^{n+2}$.
9. $x^{2n+2} \cdot 3\,x^{n+1}$.
10. $x^{n-1} \cdot x^{n+1}$.
11. $4\,x^{n-1} \cdot 3\,x^{-n+2}$.
12. $(-4\,x^n)(3\,x^{2n})(-5\,x^{2n+1})$.
13. $m \cdot m^n$.
14. $3\,x \cdot 3\,x^n \cdot 3\,x^{3n}$.
15. $2 \cdot 2^3$.
16. $2^4 \cdot 2^3$.
17. $2^8 \cdot 2^4(-2)$.
18. $3^2 \cdot 3^4$.
19. $2 \cdot 3^2 \cdot 2^4 \cdot 3^3$.
20. $(-2^3)(-2^4)(-2^5)$.
21. $12\,xy$ by $\frac{1}{2}\,xy$.
22. $.5\,x^2$ by $.4\,x^2$.

23. $4(x + y)$ by $6\frac{1}{2}(x + y)$. **25.** $.16(x+y)^3$ by $\frac{1}{2}(x+y)$.

24. $.5(x + y)^2$ by $.3(x + y)^2$. **26.** $.8(x+y)^{3a}$ by $2(x+y)^{5a}$.

27. $.24(x + y)^{4a+2}$ by $\frac{1}{2}(x + y)^{a+3}$.

28. $6(x + y + z)$ by $4(x + y + z)^2$.

29. $5\ x^2$ times $3\ x^3$ minus $2\ x^4$ times $6\ x$ equals?

30. $18\ x$ minus 3 times $5\ x$ equals?

31. 4 times $23\ x^2$ plus $8\ x$ times $5\ x$ equals what?

32. What multiplied by $.7\ x^5$ equals $.77\ x^7$?

33. $3\ a^2mn^3 \cdot 5\ a^6m^4nx$.

34. $-3\ x^2y^4z^7 \cdot 11\ x^4yz^3$.

35. $(15\ x^3y^2z)(-3\ xy^4z^5)(-2\ x^2yz^3)$.

36. $(12\ xyz^4w)(3\ x^3y^2zw^4)(-2\ x^2yzw^3)(-5\ xzw^2)$.

37. $(\frac{1}{2}\ x^3)(-\frac{1}{3}\ x^5)(12\ x^7)$.

38. $(.6\ x^2)(-.4\ x^3)(5\ x^4)$.

39. $(12\ xz^2)(-6\ x^2z^3)(-3\ x^5z^4)$.

11. Multiplication by a Polynomial. — When the multiplier is a polynomial, the multiplication is performed by the following

Rule. — *Multiply the multiplicand by each term of the multiplier in succession, placing similar terms in vertical columns. Add the partial products.*

Illustrative example I:

Multiply $x^3 - x^2 + 6\ x - 3$ by $x - 5$.

Solution:

$$\begin{array}{r} x^3 - x^2 + 6\ x - 3 \\ x - 5 \\ \hline x^4 - x^3 + 6\ x^2 - 3\ x \\ -5\ x^3 + 5\ x^2 - 30\ x + 15 \\ \hline x^4 - 6\ x^3 + 11\ x^2 - 33\ x + 15 \end{array}$$

Product.

Illustrative example II:

Multiply: $x^3 + x^2y + xy^2 + y^3$ by $x - y$.

Solution:

$$\begin{array}{r} x^3 + x^2y + xy^2 + y^3 \\ \underline{x - y} \\ x^4 + x^3y + x^2y^2 + xy^3 \\ \underline{\quad - x^3y - x^2y^2 - xy^3 - y^4} \\ x^4 - y^4 \quad \text{Product.} \end{array}$$

12. Checks of Multiplication. — Multiplication may be checked by division. The product divided by either of the quantities multiplied should give the other. A numerical check similar to those suggested in addition and subtraction is also possible. In the second illustrative example above let $x = 3$ and $y = 2$.

Then $x^3 + x^2y + xy^2 + y^3 = 27 + 18 + 12 + 8 = 65$, and $x - y = 1$.

Also $x^4 - y^4 = 81 - 16 = 65$.

Since $65 \cdot 1 = 65$, the result is correct.

13. The Parenthesis in Multiplication. — The parenthesis furnishes a convenient method of indicating a multiplication. For example $-3x(m + n)$ means that $m + n$ is to be multiplied by $-3x$, and $(a^2 + ab + b^2)(a - b)$ means that the product of $a^2 + ab + b^2$ and $a - b$ is desired.

Exercises

Perform each of the indicated multiplications:

1. $-3ax(a + x)$.
2. $2x^2(3x - 1)$.
3. $-2a^2x^2(3a^2 + 2ax - x^2)$.
4. $-mn(m^2 - mn + n^2)$.
5. $a^2x^3y(2a^2 - 3axy - 4ay^2 - x^2 + 1)$.
6. $3a^3 - 6a^2 + 7a - 3$ by $a - 1$.
7. $4a^3 + 6a^2 - 9a + 11$ by $3a^2 + 2$.

8. $(x^2 - x + 1)(x + 1)$.

9. $(3x^3 + 6x^2 - 8x + 1)(x^2 + x - 3)$.

10. $(a^3 - a^2x + ax^2 - x^3)(a^2 + ax + x^2)$.

11. Multiply $a^4 + 6a^3 - 3a^2 - a + 3$ by $a^2 + 3a - 5$.

12. Multiply $a^4 + a^2 + 1$ by $a^4 - a^2 + 1$.

13. Multiply $x^2 + y^2 + z^2 - xy - xz - yz$ by $x + y + z$.

14. Multiply $a^8 - a^4 + a^2 - 1$ by $a^4 - a^2 + 1$.

15. Simplify $(a^2 - 2a + 1)(a - 3) - a(a^3 + a + 2)$.

Solution:
$$\begin{aligned}&(a^2 - 2a + 1)(a - 3) - a(a^3 + a + 2)\\ &= (a^3 - 5a^2 + 7a - 3) - (a^4 + a^2 + 2a)\\ &= a^3 - 5a^2 + 7a - 3 - a^4 - a^2 - 2a\\ &= -a^4 + a^3 - 6a^2 + 5a - 3.\end{aligned}$$

Simplify:

16. $(x^3 + x^2 + x + 1)(x - 1) - (x^3 - x^2 - x - 1)(x + 1)$.

17. $6x^3 + (x - 1)(x^2 + x + 2) - x(x + 3)$.

18. $4(m + n)(m - n) - (m^2 + mn + n^2)(m^2 - n^2)$.

19. $(x^2 + 3x - 2)(x^2 - 2x + 3) - 3x + 6(2x^3 - 6x^2 - 8x + 1)$.

20. Multiply $1 - 6x^3 + 4x^2 - 3x$ by $-x + 4x^2 - 1$, leaving the terms in the same order as given. Then multiply again after having arranged the terms of both multiplicand and multiplier in descending powers of x.

21. Write a paragraph explaining why it is desirable to arrange the terms in order of exponents of a common letter before you multiply.

Arrange and multiply:

22. $m^2n - 6m^3 - 4mn^2 + n^3$ by $mn - n^2 + m^2$.

23. $a^4 + 3ab^3 + b^4 - a^3b$ by $b^2 - a^2$.

Multiply:

24. $x^3 - x + 2x^2 + x^4 - 1$ by $x - 1 + 2x^2$.

25. $2xy + 6x^2 - y^2$ by $-2xy - y^2 + 3x^2$.

Review

1. From what expression must $15x^2 - 6xy - 9yz$ be subtracted so as to leave $9x^2 + 8xy + 9yz$?

2. Multiply $a^x + b^y$ by $a^x + b^y$. Check if $a = 4$, $b = 5$.

3. If $a = 1$, $x = 2$, $y = 3$, and $z = 4$, evaluate

(a) $2x^2 - 5a(z - y)$.

(b) $4x(x - y) + a^2z$.

(c) $a^2z + 3(x^2 - y^2) - 2xyz$.

(d) $(x - y)(x + y) + (x + z)(x - z)$.

(e) $x^2y^2z^2 - x(a^2 - y^2)$.

4. Solve and check each of the following equations:

(a) $2.5x + 4.6x = 3.5x + 20.8$.

(b) $\frac{2}{3}x + 17 = \frac{5}{6}x - 19$.

(c) $5(4x + 5) - 6(2x + 3) = -4(x - 1)$.

5. (a) What is the length in feet of a fence around a rectangular lot 1 foot long and y yards wide?

(b) If I buy a lot for d dollars and build a house for \$8000 and sell the property so as to make p dollars, what is the selling price?

(c) What is the cost of h handkerchiefs at one fourth of a dollar each?

(d) If I lose d dollars on a piece of property that costs me \$1200, what is the selling price?

6. What is the meaning of the direction "solve an equation"?

7. Is 7.5 a root of the equation $9x - 6 = 7x + 9$? Why?

8. If a number is represented by n, what represents 25% of the number? 60% of the number? 35% of the number?

9. If an automobile can go on an average of 25 miles an hour, how far will it go in 6 hours? h hours? $\frac{1}{5}h$ hours? $.4h$ hours?

10. If c represents the cost of 1 yard of cloth, what doe
$\frac{5}{3}\,c$ represent?

11. If c represents the cost, p represents the price of on
yard, and n represents the number of yards, write a formul
for the cost of any number of yards.

12. Find the value of v in the formula $v = lwh$ whe
$l = 6.5$, $w = 1.6$, and $h = \frac{1}{4}\,w$.

13. If $x = 3$ and $y = 4$, what does $xy^2 - 3\,xy$ equal?

14. Add the following expressions and check:
$x^2 + y^4 + z^3$, $-4\,x^2 - 5\,z^3$, $8\,x^2 - 7\,y^4 + 10\,z^3$, $6\,y^4 - 6\,z^3$.

15. What expression must be added to $3\,x^2 - 5\,x - 2\,x^3 +$
to produce $3 - x^3 - 4\,x + 4\,x^2$?

14. Exponents in Division. — Since $a^2 \cdot a^3 = a^5$, it follow
that $\frac{a^5}{a^2} = a^3$ and $\frac{a^5}{a^3} = a^2$. Hence the exponent of any lette
in a quotient is equal to the exponent of that letter in th
denominator or divisor subtracted from the exponent of tha
letter in the numerator or dividend. In symbols

$$\frac{x^m}{x^n} = x^{m-n}.$$

15. Division by a Monomial Divisor. —

Rule. — *Divide each term of the dividend in succession b
the divisor, being careful to follow the rules for signs and ex
ponents already stated.*

Oral Exercises

Perform the indicated divisions.

1. $\dfrac{-10\,a^3b^6}{5\,ab^4}$.

2. $\dfrac{12\,a^5b^4x^2}{3\,a^3bx^2}$.

3. $\dfrac{14\,m^4n^3x}{-7\,m^3n}$.

4. $\dfrac{a^2b - ab^2}{-ab}$.

5. $\dfrac{x^3y^2 - xy^3 + x^4y}{xy}$.

6. $\dfrac{12\,x^3 - 6\,x^2 - 18\,x^5}{-\,6\,x}$.

7. $\dfrac{42\,a^5b^6c^4d^3}{-\,7a^2bc^4d^2}$.

8. $\dfrac{-\,120\,x^4y^3z^7 + 40\,x^2y^5z^4 - 30\,x^6y^2z^8}{-\,10\,x^2y^2z^4}$.

9. $\dfrac{-\,16\,a^{2m} - 6\,a^{6m}}{-\,2\,a^m}$.

10. $\dfrac{10x^{n+3} - 8x^{n+2} - 6x^{n+1}}{-\,2\,x}$.

11. $\dfrac{2^6}{2^3}$.

12. $\dfrac{-\,2^{10}}{2^5}$.

13. $\dfrac{-\,3^6}{-\,3^2}$.

14. $\dfrac{-\,2^8}{(-\,2)^8}$.

15. $\dfrac{3^2 \cdot 2^6}{-\,3 \cdot 2^4}$.

16. $\dfrac{-\,3^{n+1} \cdot 5^{n+6}}{3^n \cdot 5^{n+3}}$.

17. $(\frac{1}{4}\,x^2 + \frac{1}{8}\,x^3 + \frac{4}{7}\,x^6) \div \frac{1}{2}\,x^2$.

18. $.6\,x^2)\underline{1.2\,x^2 + 3\,x^3 - 6\,x^4}$.

19. $\dfrac{8(x-y) + 4(x-y) - 6(x-y)}{2(x-y)}$.

20. What must $-\,H^2K^2$ be multiplied by to produce $H^3K^4 - 6\,H^2K^3 + 5\,H^4K^5$?

21. Divide: $\dfrac{.25}{-.5}$, $\dfrac{2.75}{-.25}$, $\dfrac{125}{.25}$, $\dfrac{7.75\,x^2}{-.25\,x}$.

Divide and check:

22. $18\,x^2y$ by $9\,x^2y$.

23. $.27\,x^2y^3$ by $.03\,x^2y^3$.

24. $\frac{1}{5}\,x^4y^7$ by $\frac{1}{3}\,x^4y^7$.

25. $16\,x^5$ by $4\,x^5$ by $\frac{1}{2}$.

26. $.6\,x^3y$ by $.3\,xy^2$.

27. $3\,x^4y^{5a+b}$ by xy^{3a+b}.

28. $6\,x^{2a+b+c}$ by $3\,x^{a+2b+3c}$.

29. $2.4(x+y)^3$ by $.6(x+y)$.

30. 2^{5x} by 2^{3x}.

31. $\dfrac{-\,36(x+y)^{4a+6}}{4(x+y)^{3a+3}} = ?$

Review

1. Define and illustrate:

(a) Negative number.
(b) Coefficient.
(c) Exponent.
(d) Absolute value.
(e) An equation.

2. If $x = 6$, write the right-hand member of the equations:

(a) $.5\,x + .75\,x =$
(b) $\frac{1}{3}\,x + \frac{1}{2}\,x + \frac{1}{12}\,x =$
(c) $x^2 + 4\,x + .5\,x =$

3. If $a = 4$, $b = 2$, $c = 0$, $d = 3$, find the value of

(a) $4\,ad^2$.
(b) $2\,a^3b$.
(c) $2\,a^2b - 7$.
(d) $3\,b + 4\,a$.
(e) $\dfrac{3\,a - 2\,b}{d}$.
(f) $\dfrac{3\,a^2 + 4\,d^3}{5}$.
(g) $d^3 + \dfrac{c}{b}$.
(h) $\frac{1}{2}\,a^2b + \frac{1}{4}\,ab^2$.

4. If my debts are diminished, am I richer or poorer? Prove it.

5. Express in algebraic symbols:

(a) Five times x plus six times y.

(b) Five times the cube of x minus four times the square of y.

(c) Four x square plus five times the quantity x minus y.

(d) Six x cube diminished by twice y square.

(e) One fifth x cube plus one fourth y cube minus one sixth z square.

6. Multiply $a^4 + 3\,ab^3 + b^4 - a^3b$ by $b^2 - a^2$.

7. Add, $3\,x^{2a+3} + 5\,x^{a+2} - 6\,x^{3a-1}$; $5\,x^{3a-1} - 4\,x^{2a+3} - 6\,x^{a+2}$; $9\,x^{a+2} - 7\,x^{2a+3} - 16\,x^{3a-1}$; $12\,x^{3a-1} + 2\,x^{2a+3} - 8\,x^{a+2}$.

8. Is 2, 6, or 3 a root of $.7\,x - .9\,x + .35\,x = .45$? Prove.

16. Division by a Polynomial Divisor. — The method of dividing one polynomial by another is indicated by the following

Illustrative example I:

Divide $2x^3 - 3x^2 + 7x - 3$ by $2x - 1$.

Solution:

$$\begin{array}{l|l} 2x^3 - 3x^2 + 7x - 3 & 2x - 1 \\ \underline{2x^3 - x^2} & \underline{x^2 - x + 3} \\ -2x^2 + 7x & \\ \underline{-2x^2 + x} & \\ 6x - 3 & \\ \underline{6x - 3} & \end{array}$$

The above example is worked by the aid of the following

Rule. — (1) *Arrange both dividend and divisor according to either the ascending or descending powers of a common letter.*

(2) *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

(3) *Multiply the entire divisor by the term of the quotient just found and subtract the result from the dividend.*

(4) *Divide the first term of the remainder by the first term of the divisor to obtain the second term of the quotient.*

(5) *Multiply the entire divisor by the second term of the quotient, subtract, and so continue until the divisor is exact or the remainder is of lower degree than the divisor.*

Care should be taken to keep the remainder arranged in the same order as the dividend and divisor.

For synthetic division see page 393.

Illustrative example II:

Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

Solution:

$$\begin{array}{l|l} x^4 + x^2y^2 + y^4 & x^2 + xy + y^2 \\ \underline{x^4 + x^3y + x^2y^2} & \underline{x^2 - xy + y^2} \\ -x^3y + y^4 & \\ \underline{-x^3y - x^2y^2 - xy^3} & \\ x^2y^2 + xy^3 + y^4 & \\ \underline{x^2y^2 + xy^3 + y^4} & \end{array}$$

If there is a remainder, it should be written over the divisor in the form of a fraction, and the resulting fraction connected to the quotient by a + sign.

Illustrative example III:

Divide $x^4 + y^4$ by $x - y$.

Solution:

$$
\begin{array}{l|l}
x^4 + y^4 & x - y \\
\underline{x^4 - x^3y} & \\
x^3y + y^4 & x^3 + x^2y + xy^2 + y^3 + \dfrac{2y^4}{x - y} \\
\underline{x^3y - x^2y^2} & \\
x^2y^2 + y^4 & \\
\underline{x^2y^2 - xy^3} & \\
xy^3 + y^4 & \\
\underline{xy^3 - y^4} & \\
+2y^4 &
\end{array}
$$

17. Checks in Division. — One of the best checks in division is multiplication. The dividend is equal to the product of the quotient and divisor, plus the remainder if any. If the division is exact, a numerical check is often advantageous. Thus in Illustrative example I above if $x = 2$, the dividend equals 15, the divisor 3, and the quotient 5. $15 \div 3 = 5$.

Exercises

Divide:

1. $x^2 - 3x - 10$ by $x - 5$.
2. $6a^2 + 5a - 21$ by $2a - 3$.
3. $3a^3 + 6a^2 - 5a - 10$ by $a + 2$.
4. $6n^3 - 3n^2 - 38n + 20$ by $2n - 5$.
5. $3x^3 - 22x^2 + 16x - 3$ by $x^2 - 7x + 3$.
6. $2x^4 + x^3 - 19x^2 - x - 28$ by $2x + 7$.

7. $3x^4 - 23x^3 - 7x^2 + 42x - 10$ by $3x^2 + x - 5$.

8. $2x^5 - 9x^4 + 15x^3 - 27x^2 + 20x - 4$ by $2x^2 - 7x + 2$.

9. $x^3 - 7x^2y + 3xy^2 - 21y^3$ by $x^2 + 3y^2$.

10. $2x^4 - 5x^3y + 4x^2y^2 - 2xy^3 - y^4$ by $x^2 - xy + y^2$.

11. $x^4 - y^4$ by $x - y$.

12. What must $x + y$ be multiplied by to produce $x^4 - y^4$?

13. What must $x - 2$ be multiplied by to produce $x^5 - 32$?

14. What must $x^4 + x^2 + 1$ be multiplied by to produce $x^8 + x^4 + 1$?

15. If the product is $m^6 + n^6$, the multiplier $m^2 + n^2$, what is the multiplicand?

16. $(x^2 - 2y^2 + xy - 3z^2 + 7yz + 2xz) \div (3z + x - y) = ?$

Divide:

17. $3x^2 - 2x^4 + x^5 - x^3 - 2x - 15$ by $x^3 - 5 - 4x$.

18. $x^6 - y^6$ by $x^2 + xy + y^2$.

19. $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

20. By what must the quantity $(x^2 - 2x + 1)$ be multiplied to produce $x^6 - 2x^3 + 1$?

Divide:

21. $x^5 + y^5 + x^4 + y^4 - x^4y - xy^4 - 2x^2y^2$ by $x^3 + y^3 - x^2y - xy^2$.

22. $x^3 + y^3$ by $x - y$.

23. $8y^3 - y^2 - 5$ by $2y - 3$.

24. $9x^4 - 67x^2 + 50x + 36$ by $x^2 + x - 6$.

25. $x^5 - y^5$ by $x - y$.

26. $6x^{3a} - 25x^{2a} + 27x^a - 5$ by $2x^a - 5$.

27. $x^{3a-1} - x^{2a} + 3x^{a-1} - 3$ by $x^{a-1} - 1$.

28. $6x^{2a} - 7x^ay^b - 20y^{2b}$ by $2x^a - 5y^b$.

29. $.12x^{3a} - .09x^{2a}y^b - .2x^ay^{2b} + .15y^{3b}$ by $.4x^a - .3y^b$.

30. $2x^{\frac{5}{6}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} - 3x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y^{\frac{5}{6}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{3}}$.

31. $x^{\frac{3}{2}} + xy^{\frac{1}{2}} + x^{\frac{1}{2}}y + y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

32. $-\frac{3}{4}x^4 - \frac{5}{8}x^3 + \frac{119}{80}x^2 - \frac{3}{20}x - \frac{1}{5}$ by $\frac{3}{4}x^2 + x - \frac{4}{5}$.

33. $x^6 + \frac{5}{6}x^4 - \frac{13}{36}x^3 - \frac{1}{6}x^5 + \frac{37}{36}x^2 - \frac{17}{36}x + \frac{7}{24}$ by $x^2 - \frac{2}{3}x + \frac{1}{2}$.

34. $(2x^4 + 3x^2 - 5x + 2 - 6x^3) \div (x^2 - 3x + 1)$.

35. $(6x^3 + 4xy^2 - 13x^2y + 2y^3) \div (2x - 3y)$.

36. $(2x^4 + 6x^2 - 4x - 5x^3 + 12) \div (x^2 - x + 1)$.

37. $(4x^4 - 4x^2 + 1) \div (2x - 2x^2 + 1)$.

38. $(5x^3 - 8x^2 + 8x + 7) \div (5x - 3)$.

39. $(x^4 - x^3 - 7x^2 + 28x - 11) \div (x^2 - 4x + 7)$.

40. $(x^4 - 2x^3 - 10x) \div (x^2 - 3x + 1)$.

41. $x^6 \div (x^2 - 2x + 1)$ to four terms.

42. $(x^7 - 4) \div (x^2 - 1)$ to four terms.

Exercises — Word Problems

1. Write the following by means of algebraic symbols:

(a) Oranges are selling at 3 cents each. What is the cost of o oranges?

(b) Oranges are sold at 2 cents each and at a loss of c cents each. What was the cost of o oranges?

(c) Apples are sold at 5 cents each and c cents profit is made on each apple. What was the cost of one apple? Of a apples?

2. By how much is $4k$ more than $3h$? $3x$ more than $4y$? By how much does x exceed 5? 5 exceed x?

3. A line is 36 inches long. If a piece i inches is cut off, how much remains? If $(i + 2)$ inches were cut off? $(i - 4)$ inches?

4. Two birds fly away from a point in opposite directions, one at the rate of 35 miles per hour and the other at the rate

of 40 miles an hour. How far apart will they be in h hours? In $(h + 2)$ hours? In $(h - 6)$ hours?

5. A horse and a cow together cost \$275. If the horse cost d dollars, how much did the cow cost?

6. George threw a ball 40 feet and his brother threw one f feet farther. How far did his brother throw the ball?

7. If I buy a hat for d dollars and give a \$10 bill in payment, how much change do I receive?

8. A half quire of paper contains 12 sheets. How many sheets are there in h half quires? In x quires? In $2x$ quires? In $.5x$ half quires?

9. There are 5280 feet in a mile. How many feet in $5x$ miles? In $\frac{1}{2}x$ miles? In $.25x$ miles?

10. A book cost c cents to make, o cents for overhead, and then is sold at a profit of p cents. What is the selling price of one? Of 7? Of $9x$?

11. A man sent an article by freight. It cost him 75 cents for the freight and i cents to insure it against damage or fire, and c cents for cartage. The buyer pays x cents for cartage and s cents for its storage for 30 days. He then sells the article at a gain of g cents. What is the selling price of one? Of five?

12. The difference of two numbers is 15. The smaller number is x, what is the larger?

13. The sum of two numbers is 15. One of them is x, what is the other?

14. The product of two numbers is 15. One of them is x, what is the other?

15. The quotient of two numbers is 15. One of them is x, what is the other if there is a zero remainder?

16. A cube is i inches long. What is the total length of all of its edges?

17. A rectangle is l inches long and w inches wide. What is its area? Its perimeter? What would $2\,l(w + 3)$ represent?

18. A rectangle has an area of xy square inches. If the width is y inches, what is its length? Its perimeter?

19. A square is i inches on a side, what is its area? Its perimeter?

20. A square has an area of s^2 square inches. What is the length of a side?

21. What is the multiplier when the product is $9\,x^2y$ and the multiplicand is $3\,y$?

22. What number multiplied by $5\,x$ gives $6\,y$ as a product?

23. How much is left from a $10 bill after paying for 6 yards of kersey at k cents a yard and 8 yards of cashmere at c cents a yard?

24. Two men are 50 miles apart, and walk toward each other, one at the rate of 2 miles an hour, the other, 3 miles an hour. How far apart will they be at the end of h hours?

25. John bought 10 quarts of milk, and on his way he spilled 3 times q quarts, less 4 quarts. How many quarts had he left?

Exercises

1. The population of a certain town at the present time is P. If it gains $b\%$ of this population by births and loses $d\%$ of this population by deaths regularly each year for n years, what will represent the population at the end of that time?

2. The sides of a rectangular solid are x, $x + 2$, and $x + 3$. What expresses the volume? Find the volume when $x = 3$.

3. The area of a circle may be denoted by the formula $A = \frac{1}{2} Cr$. If $C = 2\pi r$, find the area of a circle whose radius is 10.

4. If the space s passed over by a body moving with a uniform velocity v in the time t is represented by the formula $s = vt$, what is the space traveled by:

(a) An automobile, when $v = 30$ miles per hour, in 12 hours.

(b) A bicyclist, if $v = 15$ miles per hour, and $t = 3$ hours.

(c) A train in 6 hours, when $v = 45$ miles per hour.

(d) A bird in 3 hours, when $v = 3000$ feet a minute.

5. The volume of a cylinder is denoted by the formula

$$V = \pi r^2 h.$$

If 1 cu. ft. will hold $7\frac{1}{2}$ gallons, how many gallons will a cylindrical tank 10 feet high with a radius of the base of 5 feet hold?

6. A man has a large manufacturing plant which is worth \$15,000. He is taxed on three fourths of its value at the rate of d dollars per hundred dollars. If T represents the tax, write a formula for the tax on his property.

7. The formula used to change Fahrenheit thermometer reading to Centigrade reading is $C = \frac{5}{9}(F - 32)$.

What is the Centigrade reading when the Fahrenheit reading is (a) 113°? (b) 55°? (c) 32°? (d) $-23°$?

8. The value of American exports for three different years was as follows:

One year \$4,409,000.

Five years later \$21,636,000.

One year later \$28,300,000.

Construct a graph by the use of bars to represent the above facts.

9. The value of the imports into the United States for the same years as stated in example 8 was

\$4,910,000, \$2,446,000, \$2,000,000.

Construct a graph to represent these facts.

10. The average income of a group of college graduates for ten years after graduation was

First year	\$ 700	Sixth year . . .	\$2400
Second year . . .	850	Seventh year . .	2300
Third year	1200	Eighth year . . .	2700
Fourth year . . .	1700	Ninth year . . .	3250
Fifth year	2000	Tenth year . . .	3800

Construct a curve graph to represent these facts.

11. The population of Buffalo as given in round numbers at three consecutive ten-year periods was

1900	352,000
1910	423,000
1920	506,000

Represent the above facts by a curve graph.

If we consider the average increase for ten years, continue the curve and estimate the population in 1930.

In 1950. In 1970.

12. At the same place on two successive New Year's days the temperature readings were as follows:

	Midnight	4 A.M.	8 A.M.	Noon	4 P.M.	8 P.M.
1st New Year's	8°	5°	7°	12°	14°	12°
2d New Year's	− 4°	− 6°	− 4°	0°	2°	− 1°

Draw two curves on the same chart to show the above facts.

At what hour was there the greatest difference of temperature?

13. The gross revenue of a certain telephone system was disposed of as follows:

Salaries and wages . . .	50%
Interest	19%
Surplus	6%
Taxes	5%
Material, rent, traveling .	20%

Represent the above data on a chart. Use a rectangle and let its length represent 100%.

14. The earnings and expenses of a certain railroad by years are given in the following table:

	1915	1916	1917	1918	1919	1920
Earnings	$400,000	$600,000	$680,000	$800,000	$960,000	$1,000,000
Expenses	260,000	610,000	460,000	500,000	640,000	680 000

Construct the two curves on the same diagram.

When do the two curves cross?

What does this mean?

Which curve should be at the top for practically all of the time? Why?

CHAPTER II

THE EQUATION

You have learned in your earlier work in algebra how valuable the equation is in all applied work. The ability to express relations between magnitudes in equation form and to solve the equation is the essence of algebra. You can not be too careful, therefore, in endeavoring to obtain correct ideas of this important topic.

18. An Equation is a statement that two algebraic expressions are equal. These expressions are called the *members of the equation*. Equations are of two types, namely, *Identical Equations* and *Conditional Equations*.

The identical equation or identity is true for all values of the letters concerned. Examples: $2 \cdot 3 = 6$,

$$(a + b)(a - b) = a^2 - b^2, \; 2x + 3x = \frac{10}{2}x.$$

When we desire to emphasize the identical character of an equation, we use the sign $\equiv$. This sign is read "is identical with."

The conditional equation is true only for a limited number of values of the letter concerned. Examples: $2x - 3 = 5$, which is true only when $x = 4$; and $x^2 - x = 6$, which is true only when $x = 3$ or -2.

NOTE. *All numerical equations are identities.*

19. Roots of Equations. — A root of an equation is that value of the unknown quantity which makes the equation

true. Only conditional equations have roots. In the illustrations of the preceding paragraph the root of the first equation is 4, and the roots of the second equation are 3 and -2.

You should note that when the root is substituted for the unknown quantity in a conditional equation, it is reduced to a numerical identity.

20. Simple Equation. — A simple or linear equation in one unknown quantity is an equation which contains that unknown quantity to the first power only.

21. Solution of Equations. — The solution of an equation is the process of obtaining the root or roots. In simple linear equations you will accomplish this by the aid of certain *axioms*, which are already familiar to you both in algebra and geometry. They are the axioms of addition, subtraction, multiplication, and division, respectively, and may be stated as follows:

1. If the same number be added to both members of an equation, the equation remains true.
2. If the same number be subtracted from both members of an equation, the equation remains true.
3. If both members of an equation be multiplied by the same number, the equation remains true.
4. If both members of an equation be divided by the same number (except zero), the equation remains true.

Illustrative example I: Solve $x - 5 = 10$.

Solution: Adding 5 to each member of the equation according to the first axiom, we obtain $x = 15$.

Illustrative example II: Solve $x + 6 = 8$.

Solution: Subtract 6 from each member of the equation according to the second axiom and we obtain $x = 2$.

Illustrative example III: Solve $\frac{1}{2}x = 8$.

Solution: Multiply each member of the equation by 2 according to the axiom of multiplication and we obtain $x = 16$.

Illustrative example IV: Solve $7x = 21$.

Solution: Divide each member of the equation by 7 according to the axiom of division and we obtain $x = 3$.

Illustrative example V: Solve $4x - 3 = 2x + 5$.

Solution:

$4x = 2x + 8.$	Axiom of addition.
$2x = 8.$	Axiom of subtraction.
$x = 4.$	Axiom of division.

Oral Exercises

Solve each of the following equations and state which axiom or axioms are used in each case:

1. $3x = 9.$
2. $x - 5 = 8.$
3. $x + 6 = 2.$
4. $\frac{x}{3} = 6.$
5. $2x + 1 = 5.$
6. $9x - 3 = 15.$
7. $4x = x + 6.$
8. $2x = 9 - x.$
9. $\frac{1}{4}x = 3.$
10. $2x = \frac{x}{2} + 6.$
11. $4x - 2 = 3x + 5.$
12. $5x = x - 10 - 2x.$
13. $1 + 3x = x - 11.$
14. $.5x = 25.$
15. $6x = .54.$
16. $8 = 4x.$

17. Give an illustration to show that the axiom of division fails when the divisor equals zero (0).

18. Point out the error in the following argument:

Let	$a = b.$	
Then	$a^2 = ab.$	Axiom of multiplication.
Then	$a^2 - b^2 = ab - b^2.$	Axiom of subtraction.
	$(a + b)(a - b) = b(a - b).$	Factoring.

Then	$a + b = b.$	Axiom of division.
But $a = b.$	$\therefore a + b = 2\,b.$	
Hence	$2\,b = b.$	
	$\therefore 2 = 1.$	Axiom of division.

22. Transposition is the process of omitting a term from one member of an equation and writing the same term with the *opposite sign* in the other member. *It is equivalent to the addition and subtraction axioms.*

Illustrative example I: Solve $x - 5 = 8$.

Solution: By transposing the -5 we obtain

$$x = 8 + 5 \text{ or}$$
$$x = 13.$$

This is equivalent to adding 5 to each member of the equation.

Illustrative example II: Solve $2\,x + 3 = 11$.

Solution: By transposing $+3$ we obtain $2\,x = 11 - 3$ or $2\,x = 8$ or $x = 4$.

This is equivalent to subtracting 3 from each member of the equation.

We should think of transposition, not as a process of algebra, but merely as a short way of applying the axioms of addition and subtraction. It is especially useful when several terms are to be transposed.

Rule. — (1) *To solve a simple equation containing one unknown quantity, transpose to the left member of the equation each term containing the unknown quantity, and to the right member each term not containing it.*

(2) *Collect similar terms in each member.*

(3) *Divide each member by the coefficient of the unknown quantity.*

23. Check of Solutions. — You should remember that the final test as to whether a quantity is or is not a root of an equation is whether it does or does not cause the original equation to reduce to a numerical identity when it is substituted for the unknown quantity. If it does so, it is said to *satisfy* the equation and is a root. To be a valid check the substitution must be made in the *original* equation.

24. Literal equations are equations in which some or all of the known quantities are represented by letters. It is customary to use the last letters of the alphabet to represent unknown quantities and the earlier letters to represent known quantities. This custom has been followed in this book except as otherwise stated. Formulas in which the letters used are initial letters are exceptions to this rule.

Illustrative example I:

Solve and check: $3x - 10 + x = 5 - 2x + 3$.

Solution:

Transposing, $3x + x + 2x = 5 + 3 + 10$.

Collecting terms, $6x = 18$.

$\therefore x = 3$.

Check: $9 - 10 + 3 = 5 - 6 + 3$.

$2 = 2$ (an identity).

Illustrative example II:

Solve and check: $2K - x = 5x - 10K + 6$.

Solution:

Transposing, $-x - 5x = -10K + 6 - 2K$.

Collecting terms, $-6x = -12K + 6$.

$\therefore x = 2K - 1$.

Check: $2K - (2K - 1) = 5(2K - 1) - 10K + 6$.

$2K - 2K + 1 = 10K - 5 - 10K + 6$.

$1 = 1$ (an identity).

Oral Exercises

1. What is a conditional equation? An identity? Give an example of each. Which kind of equation can be solved?

2. What condition makes $2x = 10$? $x - 1 = 6$?

Is there any value of x such that $(x + 1)^2$ will not equal $x^2 + 2x + 1$? Then $(x + 1)^2 = x^2 + 2x + 1$ is what kind of an equation?

3. What is a simple or linear equation?

4. What is a root of an equation?

5. In the equation $2x - 1 = 9$, is 3 a root? How can you tell?

6. What do you mean when you say that a certain number "satisfies an equation."

7. What is transposition?

8. State the rule concerning signs in transposition.

9. When a positive term is transposed, what axiom have you used? A negative term?

10. Why is it important to check the solution of equations?

11. Why do we always check in the original equation?

12. If you are asked to solve $a - 2y + 3m = 6m - y + 5a$, what letter do you solve for? Why?

13. What exception is there in the use of the division axiom?

14. State an equation whose solution would require the use of the multiplication axiom.

15. What is the origin of the word "algebra"? See "First Course in Algebra," page 70.

16. What value satisfies $9 = \frac{1}{3}x$? $17 = \frac{1}{2}x$? $20 = .25x$? $60 = \frac{2}{3}x$?

Exercises

Solve and check each of the following equations:

1. $6x - 10 = 2x + 14$.
2. $8 - 5x = 3x + 8$.
3. $2x - 1 + 6x = 5x - 17 + x$.
4. $3y + 2y - 5 = 15 - 2y + 4 - y$.
5. $10z - 5 = 2z + 12 + 3$.
6. $2x - 3K = 5x + 12K - 3$.
7. $6x + m - x + n = 6m + 6n$.
8. $mx - 6m^2 = 4mx + 3m^3$.
9. $6x - 2(x + 3) = 5(2x - 1)$.

Hint: First remove parentheses. $6x - 2x - 6 = 10x - 5$, etc.

10. $x^2 - (x - 1)(x + 2) = 4x + 7$.
11. $(x - 3)(x + 6) - (x + 1)(x + 4) = 6x - 10$.
12. $2x(x - 3) = (2x + 5)(x - 3)$.
13. $(3x - 1)(1 - x) + 3x^2 = 4x - (2x + 6)$.
14. $(x - 3)(x + 2) + (2x - 1)(x - 5) = (3x - 6)(x + 2)$.
15. $x^2 - (x + 1)(2x + 1) = x(2 - x)$.
16. $5(x - 3) + 3(8 - x) + 29 = 7(6 - x) + 50$.
17. $y^2 + 8y - 5 = y(y - 1) + 5(y + 3)$.
18. $3(x - 7) + 5(x + 4)(x + 2) = x^2 + 4(x + 3)(x + 1) + 10$.
19. $3[x + x(x - 3) + 4] = (3x - 2)(x + 1)$.
20. $(x - 2)(x - 4) + (x - 5)(x - 3) = 2(x - 3)(x - 5)$.
21. $(x - 2)^2 + 3(x - 4)^2 = 4(x + 3)^2$.
22. $(2x + 3)(3x - 2) = x^2 + x(5x + 3)$.
23. Does $x = 1$, 2, or 3 satisfy the equation
$(3x - 2)(x + 4) - 7 = (x - 1)(2x - 3) + x^2 + 12$?

24. What value of x satisfies

$$3(6 - x) + 4(3x + 5) + 3(6 - x) = 4(3 + x)?$$

25. Solve: $h + 30 + 7x = 8 - 6h + 6x - d + b$ for x.

26. What roots are lost if we divide each member of the following equations by the expression set at the left?

(a) $x - 4$. $\qquad x^2 - 16 = 0.$

(b) x. $\qquad x^3 + 5x^2 + 6x = 0.$

(c) $x - 5$. $\qquad x^2 - 25 = 0.$

(d) $x + 2$. $\qquad x^2 + 4x + 4 = x + 2.$

27. What extra (called extraneous) root is introduced into the equation $x = 6$ if both members are multiplied by x? By $x - 3$? By $x + 2$?

28. What root is lost if both members of the equation $x^2 - 9 = x - 3$ are divided by $x - 3$?

29. Show which of the following quantities, 2, 3, -4, is a root of the equation $(x + 4)(x - 9) = x(x + 4)$.

30. Solve $105.2 - 64x = 26x - 164.8$ for the value of x.

25. Algebraic Expression.—As a preliminary to the solution of problems it is necessary for you to be able to think in terms of algebraic symbols. You should have acquired this ability in your first course in algebra. The following exercise will give you practice in this form of expression.

Oral Exercises

1. Represent 5 consecutive whole numbers if the smallest is n; if the largest is n; if n is the middle one.

2. Represent 3 consecutive odd whole numbers if x is the smallest.

3. Why should $2n$ represent an even number rather than an odd number?

4. If John is x years old now, how old was he 5 years ago? n years ago? How old will he be 10 years hence? m years hence?

State as an equation the fact that John's age 10 years hence will be twice his age 5 years ago.

5. If I travel 30 miles an hour, how far will I go in 5 hours? In h hours? If my rate is K miles an hour, how far will I go in h hours?

6. Rate multiplied by time gives what?

State several illustrations.

$$\frac{\text{Distance}}{\text{Rate}} = ? \qquad \frac{\text{Distance}}{\text{Time}} = ?$$

7. Express 3 yards, 2 feet, 5 inches as inches.

Express the sum of y yards, f feet, and i inches as inches; as yards.

8. Express the sum of x dollars and y cents as dimes.

9. What is meant by 5%? What is 5% of \$80?

10. What is y% of \$80? y% of \$$x$?

11. If a man travels y miles in x hours, how far will he go in 1 hour? In n hours?

12. If a quantity of grain will last n cows a week, how long will it last 1 cow? m cows?

13. A man bought goods amounting to d dollar x dimes, and y cents. He presented a check for k dollars. How many cents change did he receive?

14. If I exchange k dozen eggs for y yards of cloth worth x cents per yard, what is the price of the eggs per dozen?

15. Does 23 mean $2 \cdot 3$? Does it mean $2 + 3$? Does it mean 2 tens + 3 units or $10 \cdot 2 + 3$?

16. How would you write the number whose left-hand digit is x and whose right-hand digit is y?

17. Write the number expressed by the same digits reversed.

18. x yards per second is how many miles per hour?

19. If the greater of two numbers is 1 more than twice the smaller, and the smaller number is x, what is the greater? If the greater number is x, what is the smaller?

20. In an orchard, there are a apple trees and p pear trees. How many trees are there in all?

21. An organ grinder received a cents at one house, b cents at another house, and c cents at a third house. What was the value of his money in dimes?

22. There are 5280 feet in a mile. How many feet does a boy walk if he walks m miles and 575 feet? How many miles?

23. How many dollars will a man save in w weeks if he saves 50 cents a week?

24. If your pulse beats 4320 times in one hour, how many times will it beat in d days of 24 hours each?

25. A clerk received \$20 a week. If he spent \$$x$ for board and \$$y$ for other expenses, how many dollars did he save in w weeks?

26. Problems. — The ability to solve verbal problems is a gauge of your mastery of algebra and of your ability to apply it. It is therefore very valuable, especially for the more advanced pupil.

Since each problem is different from the others, no definite rules for the solution of problems can be laid down. The following suggestions are offered:

1. Read the whole problem very carefully and try to understand its meaning thoroughly before you attempt to solve it.

2. Represent the quantity you are to find by x or some other convenient letter.

3. If you are to find more than one quantity, represent the smallest one by x, and represent the related quantities in terms of x.

4. Express in symbols as an equation that which the problem states in words.

5. In checking your solution, always check in the problem rather than in the equation.

Exercises

1. Find 3 consecutive numbers whose sum is 17 more than the largest number.

2. Find 3 consecutive odd numbers whose sum is 5 more than twice the next consecutive odd number.

3. The greater of two numbers is 5 less than twice the smaller, and their sum is 28. What are they?

4. In thirty years James will be 8 times as old as he was 12 years ago. How old is he now?

5. Henry's father is now twice as old as he is; 16 years ago he was four times as old. How old is each now?

6. Two men start from points 176 miles apart and travel toward each other until they meet. Find the rate of each if one travels two miles an hour faster than the other and they meet in 8 hours.

7. A marksman hears a bullet strike the target 1650 feet away in $4\frac{1}{2}$ seconds. Find the rate of the bullet in feet per second if the velocity of sound in air is taken as 1100 feet per second.

8. 6 hours after a boy starts from a certain point on a bicycle at the rate of 12 miles an hour he is followed by a man with an automobile and is overtaken in 8 hours. What is the rate of the automobile?

9. How far may a man walk into the country at the rate of 4 miles an hour, if he rides back at the rate of 20 miles an hour and must be back in 8 hours?

Hint: Let x = number of hours walking out.

10. Mary is 24 years old. Mary is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

11. A man was hired for 30 days at \$3 per day and board for every day that he worked. Each day that he was idle he was to pay \$1 for his board. At settlement he received \$66. How many days was he idle?

12. Mr. Jones had 60 cattle and Mr. Brown had 45. Mr. Jones bought a certain number from Mr. Brown, after which he had 6 times as many cattle as Mr. Brown. How many did he buy?

13. A man bought 60 bushels of potatoes for \$104. For one kind he paid \$1.80 a bushel, and for the other kind he paid \$1.60 a bushel. How many bushels of each kind did he buy?

14. I have twice as many dimes as nickels and 10 more quarters than the number of dimes and nickels together. If I have \$8.50 in all, how many coins of each kind have I?

15. Eight boys agreed to buy a tent, but two of them were unable to pay, so that each of the others had to pay \$4 more than he had expected to pay. What was the cost of the tent?

16. What number is that, from which 6 being subtracted, and the remainder multiplied by 11, the product will be 121?

17. After A had loaned 10 dollars to B, he lacked 8 dollars of having as much money as B; and together they had 60 dollars. How much had each at first?

18. A person has 6 hours at his disposal. How far may he ride into the country at 8 miles an hour, that he may return home in time, walking back at the rate of 4 miles per hour?

19. What number must be added to each of the numbers 2, 3, 6, 5 in order that the sum of the squares of the first two numbers may be 83 less than twice the product of the last two?

20. A room is 5 feet longer than its width, and if we decrease the length by 3 feet and increase the width by 4 feet, the area will be increased by 16 square feet. What are the dimensions of the room?

21. If we increase each side of a square by 3 feet, we shall increase the area by 33 square feet. Find a side of the original square.

22. If we decrease each side of a square by 5 feet, we shall decrease the area by 45 square feet. Find the side of original square.

23. The sum of twice one acute angle of a right triangle and three times the other acute angle is 192°. How many degrees in each of the acute angles?

24. The difference of two consecutive angles of a parallelogram is 20°. How many degrees in each angle?

25. If we increase two opposite sides of a square by 4 feet and the other two opposite sides by 6 feet, we shall then have a rectangle whose area is 74 square feet more than the square. Find a side of the square.

26. The first angle of a triangle is 4 times the result of decreasing the second angle by 15°. The third angle is 3 times the second angle. How many degrees in each angle?

27. A man's property is insured in three different companies. The first company insures it for $10,000. The second company insures it for $4000 more than twice the

amount of insurance in the third company. If the property is insured for $29,000, how much is carried in the second and third companies?

28. A man has two investments totaling $10,000. On one he receives 8% interest and on the other he receives 4% interest. The 8% investment yields an annual income of $320 more than the 4% investment. How much has he invested at each rate?

29. At a certain baseball park the reserved seats cost $1 and the other seats were 50 cents each. If 3600 tickets were sold in all and the total amount received was $2225, how many tickets of each kind were sold?

30. Fifty-five coins, consisting of dollars and half dollars, amount to $36. How many coins of each kind are there?

31. A and B have $50 between them. A wins from B as much as A had originally, and he then has three times as much as B. How much had A at first?

32. If one number exceeds another number by 3 and the square of the larger exceeds the square of the smaller by 99, what are the numbers?

33. It is claimed that 200 times as many lives were lost in an earthquake in Japan in 1923 as were lost in an earthquake in San Francisco in 1906. If 100,500 lives were lost in both, how many were lost in each?

34. If the product of two consecutive even numbers is increased by 28, it will equal the square of the larger number. Find the two numbers.

35. An automobile leaves New York for Chicago on a road paralleling a railroad track at a rate averaging 26 miles per hour. Six hours later an express train leaves for Chicago, averaging 50 miles per hour. In how many hours will the train overtake the automobile?

36. A new U. S. dirigible is said to be able to make 60 miles an hour and an airplane is making 80 miles per hour. If the airplane is 60 miles behind the dirigible, in how many hours will it overtake the dirigible?

37. In the olden days a pony express, traveling at the rate of 60 miles a day, had been gone 5 days, when a second pony express was sent after them, traveling at the rate of 75 miles per day. In what time would the one overtake the other?

38. A man starts for a distant point at 6 A.M. at the rate of 4 miles per hour. Six hours later another man starts from the same place in the same direction at the rate of 20 miles an hour. When will the second man overtake the first man?

39. A man paid his bill of \$16.75 with quarters and half dollars. If he had 60 coins in all, how many had he of each kind?

40. A man invests \$1200, part at 5% and part at 4% per annum, and his total income is \$55.50. How much did he invest at each rate?

Exercises — Review

1. Add $6x^2 - 5xy + 4z$, $12x^2 + 6xy - 4z$, $13xy$, $3x^2 - 14z$, $7xy - 17z$.

2. Subtract the product of $-7x^2$ and $4yz$ from the product of $-6yz$ and $-3x^2$.

3. Multiply $4x^2 + 6.3x - 7$ by $.2x - 3$, and check.

4. Divide $x^{3a+2} - 2x^a y^{a+1} - x^{2a+2}y^{2a} + 2y^{3a+1}$ by $x^a - y^{2a}$, and check.

5. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y) = ?$

6. Subtract $bx^2 + h$ from 0 and add that result to $ax^2 + h$.

7. If I receive d dollars my first year and a regular annual increase of x dollars thereafter, what is my salary after y years? Write your answer as a formula, letting S represent the last salary.

8. Given $d = \$1200$, $x = \$100$, $y = 12$ years. Find S in the above formula.

9. Express each of the following as equations:

(a) The sum of 15 and h equals $3x$.

(b) The double of x plus one fourth of y equals 75.

(c) Five subtracted from $3x$ gives the same result as 9 subtracted from x.

(d) Four times the square of a number one larger than x exceeds the number by 7.

10. Solve $1.2x - .2(.18x - .05) = .4x + 8.9$.

11. A father's age is three times that of his son, and in 10 years it will be twice as great. How old is each?

12. Prove that

$$(a+2)(b+2) - 4(a+1)(b+1) + 6ab - 4(a-1)(b-1) + (a-2)(b-2) = 0.$$

13. Show that if $x = y + .3$, then $x^2 + x$ will equal $y^2 + 1.6y + .39$.

14. What is the interest on \$5000 for y years at $r\%$ simple interest?

15. Simplify $4x^2 - [.5x^2 - \{.7x^2 - (.8x^2 - .005)\}]$.

16. Solve and check:

$$.3x - 2(.06x - 5) = 28.$$

17. The difference of the squares of two consecutive even numbers is 48. Find the numbers.

18. A man has \$5.95 in quarters and dimes. He has 31 coins in all. How many has he of each kind?

19. If s denotes the size of a shoe and f denotes the length of the man's foot, form a table and draw a graph to show the relation of the sizes from 6 to 12 if $f = \frac{s}{3} + 8$.

20. If the tickets to an entertainment cost 50 cents for adults and 25 cents for children, and 200 tickets are sold amounting to \$80, how many of each kind are there?

21. Which of the following are identities?

(a) $\dfrac{x^2 - y^2}{x + y} = x - y.$

(b) $(x + y)^2 = x^2 + 2xy + y^2.$

(c) $x^2 - 3x + 2 = (x - 1)(x - 2).$

(d) $x^2 - 4x = (x - 8)(x + 8).$

(e) $x^2 = (x + 4)(x - 7).$

What is an identity? An equation?

22. Make six graphs from the following information. Each chart is to contain the same kind of information about each salesgirl.

	Mary	Lucy	Anna	Ruth	Agnes
Number of customers .	48	36	54	102	86
Number of buying customers	44	30	27	84	85
Number of sales . . .	85	69	86	101	94
Number of returned sales	14	0	18	12	0
Value of sales . . .	\$258	\$315	\$225	\$198	\$98
Value returned sales .	\$70	0	\$69	\$35	0

What might you learn from these facts?

23. Business men often go to great expense to find out which is the best location for a new store. One method employed is to make a count of the number of persons passing certain corners for a certain interval of time (say 15 minutes) at various times of the day. From the following facts, con-

struct curve charts to represent these numbers, and then determine what you can from them.

Hours	Corner #1 Count	Corner #2 Count	Corner #3 Count	Corner #4 Count
8–10	111	120	210	125
10–12	193	202	284	269
12– 2	360	390	330	250
2– 4	450	354	375	148
4– 6	385	480	535	316

24. Express the equation and solve it if $37\frac{1}{2}\%$ of a number equals 81.

25. If $x + .25\,x = 1.25\,x$, then $1.25\,x$ is said to represent x increased by 25% of itself. What do each of the following represent: $1.08\,x$, $1.75\,x$, $1.66\frac{2}{3}\,x$, $1.5\,x$, $x - .25\,x$, $.75\,x$, $.8\,x$?

26. If $8 \neq 10$, make it a true statement. Do the same for $6\,x \neq 9\,x$. For $9\,y \neq 15\,y$.

27. If t represents tens and u represents units, what does $5\,t + 8\,u$ represent? $8\,t + 3\,u$? $12\,t + u$?

CHAPTER III

PRODUCTS AND FACTORS WITH APPLICATIONS

PROFICIENCY in algebra requires not only that you should be able to multiply and divide polynomials, but that you should be able to do this mentally in simple cases. It is also necessary to recognize a product when you see it, and to be able to break it up into the factors of which it is composed. Without this ability much of the later work with fractions and equations would not be possible.

27. The Factors of a product are those quantities which when multiplied together produce that product. A quantity is said to be *prime* when it has no factors except itself and 1. *Prime factors* are factors which are themselves prime.

Thus the factors of 24 are 4 and 6 or 2 and 12 or 3 and 8, but these sets of factors are not prime, since 6, 12, and 8 can all be factored further. The prime factors of 24 are 2, 2, 2, and 3. In all work in factoring prime factors are required.

28. Product of a Polynomial by a Monomial. — This case has been discussed in section 10, but is included here for convenience.

Exercises

Perform each of the indicated multiplications:

1. $-2\,a^2(a^2 - 3\,ab - 6\,b^2)$.
2. $5\,ab(a^3 - a^2b + 3\,b^2)$.
3. $m^2n^2p(2\,m^2 - 6\,mnp + 5\,np^2)$.
4. $6\,x^2z(3\,x^2 - 2\,xyz - 4\,yz^2)$.

5. $-21\,m(m^2 - lmnp - 6\,np^2 + lp^3)$.

6. $8\,ap^2(2\,a^2p - 5\,ar + 6\,pr^2)$.

7. $x(x + y + \frac{1}{4}\,y^2)$.

8. $.8(x^3 - 16\,x + 24\,x - 5)$.

9. $2\,HK(H^2 + 5\,HK - 6\,K^2)$.

10. $2\,ax^2(10\,ax - 6\,a + 4\,a^2 - a^3)$.

11. $-16\,ab(ab - \dfrac{3\,c}{4} + \dfrac{3\,x^2y^2}{b} - 4\,abc^2)$.

12. $.6\,x^n(1 - 1.2\,x^n + 1.8\,x^{2n})$.

13. $4.8\,x^ny(x^{2n} - 9.6\,x^{3n}y^4 + 14.4\,x^{4n}y^5)$.

14. $\frac{1}{2}\,xy(2\,xy^2 - 6\,x^2y - 12\,x^3)$.

15. Simplify $6 + 8(.4 - .05)$.

16. Simplify $7 + 9(.8 - 0 + .02)$.

17. $.3(x^2 - .12\,x) = ?$

18. $\frac{2}{5}\,\pi(R^2 - r^2) = ?$

19. $x^{n+1}(x - 3\,x^n + 6\,x^{n-1}) = ?$

20. What is the value of $x^n(x^n - 3\,x^{2n} - 5\,x^{3n})$ when $x = 3$ and $n = 2$?

29. Expressions Having a Monomial Factor. — The expressions of this exercise are such as arise from a multiplication of the type found in the preceding exercise. Since the monomial multiplier multiplies each term of the polynomial, it can be recognized in each term of the product and removed as a factor.

Example. Factor $6\,a^3x^4 - 15\,a^2x^2y + 21\,a^5x^5y^2$.

Solution: Inspection of the terms of this polynomial shows that $3\,a^2x^2$ is contained in each. Hence it is a factor. We divide the polynomial by this monomial factor, term by term, to obtain the polynomial factor.

We obtain $3\,a^2x^2(2\,ax^2 - 5\,y + 7\,a^3x^3y^2)$.

Exercises

Factor each of the following:

1. $2\,p^2 - 6\,pt - 8\,p^3q - 16\,pq^3$.
2. $a^4x^3p - 3\,a^3x^2p^2 - 4\,a^2x^5p - 5\,a^3x^2p^6$.
3. $21\,x^2p^2y^2 - 49\,x^3p^2y^3 - 63\,x^2p^3y^4$.
4. $4\,mn - 8\,m^2n^2 - 16\,mn^3 + 12\,m^4n^2$.
5. $-3\,a^2b^3c^5d^8 - 15\,a^3b^4c^7d^4 + 21\,a^4b^2c^7d^5e$.
6. $33\,x^2p^2z^2 - 66\,x^3p^3z + 77\,x^5p^6z^2 - 11\,x^6p^8$.
7. $16\,x^2(m+n)p^2 - 12\,xy(m+n)^2p^3 - 20\,x^3y^3(m+n)^3p$.
8. $17\,x^3(x-y)^3 + 51\,x^5(x-y)^2 - 85\,x^2(x-y)^4$.
9. $4\,x(a+b+c) + 8\,x^2(a+b+c) - 12\,x^5(a+b+c)$.
10. $5\,x^3 - (6\,x^2 + 4\,x^4 - 9\,x^5)$.
11. $1.5\,x^6 + 3\,x^4 - 2\,x^3 + 3.5\,x^2 - 4.5\,x + .5\,x)$.
12. $3\,x^{2n+2} + 12\,x^{3n+3} - 18\,x^{4n+4} + 24\,x^{5n+5}$.
13. $\frac{1}{4}\,x^3 - \frac{1}{8}\,x^2 + \frac{3}{4}\,x^5$.
14. $\frac{1}{3}\,x(a+b) + \frac{1}{3}\,y(a+b)$.
15. Compute the value of $189\,\pi - 56\,\pi$ by factoring when $\pi = \frac{22}{7}$.
16. Collect the coefficients of x and y in
$$2\,ax + 2\,ay + 3\,x + 3\,y.$$
17. Collect the coefficients of y^3 and y in
$$3\,y^3 + 4\,y - 5\,y^3 + 7\,xy.$$
18. Compute 56 times \$4.43 + 44 times \$4.43 − 50 times \$4.43.

30. Binomial Factors by Grouping. — Let it be required to factor $l(x+y) - n(x+y)$.

Solution: By means of parentheses this has been grouped into two terms. $(x+y)$ is now contained in each term. If

we divide the given binomial by $(x + y)$, term by term, we obtain as the other factor $(l - n)$.

Hence $l(x + y) - n(x + y) = (x + y)(l - n)$.

Usually, however, the grouping is left for the pupil.

Illustrative example I:

Factor: $ax + ay - 2x - 2y$.

Solution:

Group in parentheses: $(ax + ay) - (2x + 2y)$.

Remove a from 1st ( ) and 2 from the 2d ( ): $a(x + y) - 2(x + y)$.

Remove binomial factor $x + y$. We then obtain

$$(x + y)(a - 2).$$

Illustrative example II:

Factor $2bx^2 - 9y + 3x^2 - 6by$.

Solution:

Rearranging terms: $2bx^2 + 3x^2 - 6by - 9y$.

Grouping: $(2bx^2 + 3x^2) - (6by + 9y)$.

$x^2(2b + 3) - 3y(2b + 3)$.

$(2b + 3)(x^2 - 3y)$. *Ans.*

Exercises

Factor each of the following, and check as directed by your teacher.

1. $x(l - m) + y(l - m) + z(l - m)$.
2. $2(a + b + c) - x(a + b + c) + 3y(a + b + c)$.
3. $ab(l - k) + 3c(l - k) + 6d(k - l)$.

Hint: $6d(k - l) = -6d(l - k)$.

4. $3lx - 6ly - kx + 2ky$.
5. $2mn + n^2 + 4lm + 2ln$.
6. $4pr - 3qs + 3ps - 4qr$.

7. $15x - 10px + 14py - 21y$.

8. $lrh + lk - mnk - rhmn$.

9. $x^3 - z - x^2z + x$.

10. $1 - 3c - 3cd + d$.

11. $x^4 - x^3 - x + 1$.

12. $e^2d^2 + e^3 + d^3 + ed$.

13. $mn(a + b) + (c + d)(a + b)$.

14. $(a + b)(c - d) - (a - b)(c - d)$.

15. $6(x + y) - 1.2(x + y)a$.

16. $HK(a + b)y - 3HK(a + b)z$.

17. $ax + 2y - 2a - yx$.

18. $ax + bx + a^2 - b^2$.

19. $H^2 - HK^2y^2 + HM^2y^2 - K^2M^2y^4$.

20. $x^{a+b} + x^ay^a + x^by^b + y^{a+b}$.

21. $a^2 - 1 + 3a^3 - 3a$.

22. $x^3 - 2x^2 + 4 - x^2$.

23. $2ax - bx + cx + 2ay - by + cy$.

24. $x^{3k} - 3x^{2k} + 3x^k - 1$.

25. $x^2 - y^2 - 4hk(x + y)$.

26. $a^{3x} + a^{2x} + a^x + 1$.

27. $(x - 3)^3 + 4(x - 3)^2$.

28. $x + 3 + (x^2 - 9)$.

29. $(3x - 4)^2 + (6x^2 - 17x + 12)$.

30. $\dfrac{(a + b)^2 - (a^2 - b^2)}{a + b} = ?$

31. Products of Binomials Having Similar Terms. — Let it be required to multiply $2x + y$ by $3x - 2y$. The work is shown here for reference. By inspection of the product we note that the first term, $6x^2$, is the product of the first terms of the given binomials, $2x$ and $3x$. The last term of the product, $-2y^2$, is the product of the last terms of the binomials, $+y$ and $-2y$. The middle term of the product, $-xy$, is the sum of the products of $2x(-2y)$ and $3x \cdot y$ or

$$\begin{array}{r} 2x + y \\ 3x - 2y \\ \hline 6x^2 - xy - 2y^2 \end{array}$$

the sum of the *cross products*. The work may also be arranged as follows:

$$(2x+y)(3x-2y) = 6x^2 - xy - 2y^2,$$

the cross products being indicated by the curved lines.

Exercises

In like manner multiply

1. $(x - y)(7x - y)$.
2. $(3x + 2y)(5x - 4y)$.
3. $(a + 3)(2a - 7)$.
4. $(a + 3b)(3a - b)$.
5. $(4x + 2)(9x - 8)$.
6. $(3a + 7b)(9a + 2b)$.
7. $(a - 2b)(a - 2b)$.
8. $(3x + 2m)(4x - 7m)$.
9. $(a - 6)(a - 6)$.
10. $(3x + 11)(3x - 11)$.
11. $(4a^2 - 3)(2a^2 + 5)$.
12. $(2a + 3y)(6a - 5y)$.
13. $(6m + 5a)(3m - 2a)$.
14. $(7b + 3)(b - 11)$.
15. $(a - 6b^2)(3a + 2b^2)$.
16. $(3x + 7y)(3x + 7y)$.
17. $(x - .4)(2x + .7)$.
18. $(5x + 3y)(3x - 7y)$.
19. $(2x - 5y)(5x - 4y)$.
20. $(2x^a + 3y^b)(3x^a - 4y^b)$.
21. $(5 - H)(H + 3)$.
22. $(-4x^3 + 5)(2x^3 - 3)$.
23. $(30 + 4)(30 + 1)$.
24. $(40 - 2)(20 + 6)$.

25. 23 dozen eggs at 42 cents a dozen by the above method would cost what?

26. $[(a + b) + 5x][(a + b) - 3x]$.
27. $[(x + y) + 3][(x + y) - 7]$.
28. $(.5 + 3x)(.4 - 9x)$.
29. $(.3 + .2x)(.4 - .3x)$.
30. $(.03x + .05y)(.02x - .03y)$.

32. **Perfect Trinomial Squares.** — You will notice that in examples 7, 9, 16 of the preceding exercises the two quantities

to be multiplied are alike. The resulting trinomial is, therefore, a perfect square. A product of this type deserves special study.

For example:

$(2x - 3y)(2x - 3y)$ or $(2x - 3y)^2 = 4x^2 - 12xy + 9y^2$.

$4x^2$ is the square of $2x$.

$9y^2$ is the square of $-3y$.

$-12xy$ is twice the product of $2x$ and $-3y$.

It follows that *a trinomial is a perfect square when and only when two of its terms are positive perfect squares, and the third term is twice the product of the square roots of the square terms.*

Exercises

Write the result of each of the following:

1. $(x + 3)^2$.
2. $(2x - 5)^2$.
3. $(4x + 3)^2$.
4. $(8x - 3)^2$.
5. $(x - 2y)^2$.
6. $(2x + 3y)^2$.
7. $(8x - 5y)^2$.
8. $(7x - 6y)^2$.
9. $(x - 11y)^2$.
10. $(m + 6n)^2$.
11. $(3p - 4q)^2$.
12. $(lmn - 3pq)^2$.
13. $(2x - 5y)^2$.
14. $(.5x + y)^2$.
15. $(.6x + .2y)^2$.
16. $(5x + .03y)^2$.
17. $(20 + 5)^2$.
18. $(40 + 7)^2$.
19. $(a^x + 1)^2$.
20. $(x^a + y^b)^2$.
21. $(x^{2b+1} + 3)^2$.
22. $(x^{3a+1} + y^b)^2$.
23. $(HK - 3M)^2$.
24. $(\frac{1}{2}x + 3)^2$.
25. $(\frac{1}{3}x - \frac{1}{2}y)^2$.
26. $[(x + y) + 2]^2$.
27. $(x + 5)^2 + (x + 3)^2 = ?$

Of what binomial is each of the following a perfect square?

28. $4m^2 + 4mn + n^2$.
29. $a^2 - 6ab + 9b^2$.
30. $16z^2 - 24z + 9$.
31. $x^2 - 12px + 36p^2$.
32. $9a^4 + 42a^2b^2 + 49b^4$.
33. $.09x^2 + .12xy + .04y^2$.
34. $9x^{2a} + 6x^ay^b + y^{2b}$.
35. $4x^2 - .12xy + .0009y^2$.

36. $\frac{1}{4}x^2 + \frac{1}{5}xy + \frac{1}{25}y^2$.

37. $.16\,x^2 - \frac{.8}{3}xy + \frac{1}{9}y^2$.

38. $(a+b)^2 + 2\,c(a+b) + c^2$.

39. $9 + 78 + 169$.

40. $.09 + .6(x+y) + (x+y)^2$.

Supply the missing term so that each of the following trinomials shall be a perfect square:

41. $x^2 - 12\,x + (\ )$.

42. $9\,m^2 + 6\,mn + (\ )$.

43. $(\ ) - 4\,xy + 4\,y^2$.

44. $(\ ) + 44\,m^2n + 121\,n^2$.

45. $4\,x^2 - (\ ) + 9\,z^2$.

46. $16\,x^2y^2 + (\ ) + 1$.

47. $121\,a^2 + (\ ) + 49\,b^2$.

48. $25\,p^2 - (\ ) + 64\,q^2r^2$.

49. $x^2 - .6\,x + (\ )$.

50. $(\ ) + \frac{1}{7}xy + \frac{1}{49}y^2$.

51. $\frac{1}{9}x^2 + \frac{1}{3}xy + (\ )$.

52. $(a+b)^2 + (\ ) + (x+y)^2$.

33. Square of Any Polynomial. — An extension of the above process enables us to square any polynomial. The following is the

Rule. — *To square any polynomial write the square of each term separately, and twice the product of every term by every other term.*

In order that no products be missed it is best to make them in a definite order as indicated by the curved lines.

Note that the signs of each squared term is +, and that the signs of the other terms are determined by the usual law of signs in a product.

Illustrative example I:

$$(a - b + 2)^2 = a^2 + b^2 + 4 - 2\,ab + 4\,a - 4\,b.$$

Illustrative example II:

$$(2x + 3y - 5m + 6) = 4\,x^2 + 9\,y^2 + 25\,m^2 + 36 + 12\,xy - 20\,xm + 24\,x - 30\,ym + 36\,y - 60\,m.$$

Exercises

Expand each of the following:

1. $(a + b - c)^2$.
2. $(2a + 3b + 1)^2$.
3. $(x - 2y - 3z)^2$.
4. $(3x + y - 5)^2$.
5. $(4a - 3m + 2n)^2$.
6. $(x - 5y + 3z)^2$.
7. $(a + b - c + 2)^2$.
8. $(2a - 3b + 4c - 5)^2$.
9. $(a^n + b^m + c^r)^2$.
10. $(\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4})^2$.
11. $(.3x + .5y + .2z)^2$.
12. $(2x - 3y + 4)^2$.
13. $(x + y + z)^2 - (x - y - z)^2 = ?$
14. $(2x + 3y)^2 + (2x + y + z)^2 = ?$
15. Prove by the use of the above method that $(5 + 10 + 15)^2$ gives the same result as 30^2.

34. Factors of Quadratic Trinomial. — A quadratic trinomial results from the multiplication of binomials having similar terms, as in the exercises on page 57.

Illustrative example I: Factor $x^2 - 3x - 10$.

Solution: First write two adjacent parentheses as

$$(\quad)(\quad)$$

Since x^2 is the product of the first terms of these binomials, they must be x and x. We therefore write

$$(x \quad)(x \quad)$$

Since -10 is the product of the last terms of these binomials, they must be 5 and -2, or -5 and 2, or 10 and -1 or -10 and 1.

But the middle term is the sum of the cross products, and only -5 and 2 give the correct middle term. Hence the factors are $(x - 5)(x + 2)$.

Occasionally several trials may be necessary to obtain the factors which give the correct middle term. Usually, how-

ever, a little experience and good judgment will enable you to eliminate most of the incorrect trials.

If the last term of the trinomial is negative, what is known of the signs in the binomial factors? What is known if the sign of the last term of the trinomial is positive?

Illustrative example II:

Factor $12\,x^2 + 7\,xy - 10\,y^2$.

Solution: $12\,x^2 + 7\,xy - 10\,y^2 = (4\,x + 5\,y)(3\,x - 2\,y)$

Since $12\,x^2 = 4\,x \cdot 3\,x$

and $-10\,y^2 = 5\,y(-2\,y)$

and $7\,xy = 5\,y \cdot 3\,x + 4\,x(-2\,y)$.

Check: $(4\,x + 5\,y)$ times $(3\,x - 2\,y)$ gives the original expression.

Exercises

Factor each of the following and check as directed:

1. $x^2 - x - 6$.
2. $x^2 - 5\,x + 6$.
3. $x^2 - 5\,x - 6$.
4. $m^2 - 2\,m - 8$.
5. $n^2 - 10\,n + 16$.
6. $r^2 - 6\,r + 9$.
7. $s^2 - s - 12$.
8. $t^2 + t - 20$.
9. $x^2 + 8\,x - 20$.
10. $x^2 + 19\,x - 20$.
11. $x^2 - 21\,x + 20$.
12. $y^2 - 6\,yz + 8\,z^2$.
13. $x^2 - 4\,nx + 4\,n^2$.
14. $x^2 - 3\,xy - 10\,y^2$.
15. $x^2 - 6\,xy + 5\,y^2$.
16. $x^2 + 10\,xy + 16\,y^2$.
17. $6\,x^2 - 5\,xy - 6\,y^2$.
18. $2\,x^2 - 9\,xy + 4\,y^2$.
19. $8\,a^2 - 6\,ab - 5\,b^2$.
20. $6\,a^2 - 19\,ab + 3\,b^2$.
21. $12\,m^2 - 25\,mn + 12\,n^2$.
22. $10\,x^2 - 31\,xy + 15\,y^2$.
23. $3 - m - 10\,m^2$.
24. $1 + 3\,ab - 10\,a^2b^2$.
25. $x^2 - 8\,xy + 16\,y^2$.
26. $.72\,x^2 + 10.5\,xy + 12\,y^2$.
27. $\frac{1}{6}\,x^2 - \frac{5}{36}\,xy - \frac{1}{6}\,y^2$.
28. $2\,H^2 + 5.8\,HK - .6\,K^2$.

29. $2(a+b)^2 - 7(a+b)(x+y) - 15(x+y)^2$.

30. $(x-y)^2 - 7(x-y) + 12$.

31. $(x+y)^2 - 8(x+y) + 15$.

32. $6\,x^{2a-4} - 14\,x^{a-2}y + 4\,y^2$.

33. $12\,x^{2a+6} - 31\,x^{a+3}y^{a-1} + 20\,y^{2a-2}$.

34. $a^2 + 4\,b^2 + 9 + 4\,ab + 6\,a + 12\,b$.

35. Arrange and factor $7\,x + 2\,x^2 + 6$.

35. Product of Sum of Two Quantities by Their Difference. — If the binomials $a+b$ and $a-b$ are multiplied by the method of section 31, the middle term will be found to disappear, and we obtain

$$(a+b)(a-b) = a^2 - b^2.$$

The product is in this case a binomial rather than a trinomial. This binomial is the difference of the squares of a and b. Hence,

Rule. — *The product of the sum and difference of the same two quantities is the difference of their squares.*

You should note that the difference of the squares is written in the same order as the difference of the quantities.

Exercises

Write the product of

1. $(a+x)(a-x)$.

2. $(a+2\,b)(a-2\,b)$.

3. $(cx+2\,b)(cx-2\,b)$.

4. $(x+y^2)(x-y^2)$.

5. $(3\,x+6)(3\,x-6)$.

6. $(7\,x^2+5)(7\,x^2-5)$.

7. $(10\,x+3\,y)(10\,x-3\,y)$.

8. $(3\,ab+4\,x^2)(3\,ab-4\,x^2)$.

9. $(10\,mn^3+3)(10\,mn^3-3)$.

10. $(10+3)(10-3)$.

11. $23 \cdot 17$.

12. $(100+5)(100-5)$.

13. $105 \cdot 95$.

14. $102 \cdot 98$.

15. $84 \cdot 76$.

16. $63 \cdot 57$.

17. $91 \cdot 89$.

18. $32 \cdot 28$.

19. $[(a+b)+c][(a+b)-c]$.

20. $[(x-2y)+3z][(x-2y)-3z]$.

21. $[4x+(m-n)][4x-(m-n)]$.

22. $(3x+y-z)(3x-y+z)$.

23. $(4m+3n+y)(4m+3n-y)$.

24. $(x-3y+8)(x+3y-8)$.

25. $[(x+y)+(m+n)][(x+y)-(m+n)]$.

26. $(.5x+6y)(.5x-6y)$.

27. $(3.2a+2.3b)(3.2a-2.3b)$.

28. $(\frac{1}{2}x+\frac{3}{4}y)(\frac{1}{2}x-\frac{3}{4}y)$. 29. $(.4x+\frac{5}{6}y)(.4x-\frac{5}{6}y)$.

30. What is the cost of 74 pounds at 66 cents a pound?

36. Difference of Two Squares. — A binomial, which is the difference of two squares, may be factored into two factors one of which is the *sum* and the other the *difference* of their square roots.

Illustrative example I:

$$81a^2-64b^2c^8=(9a+8bc^4)(9a-8bc^4).$$

Illustrative example II:

$$(a-5)^2-x^2=(a-5+x)(a-5-x).$$

Illustrative example III:

$$m^4-(px^2-q)^2=(m^2+px^2-q)(m^2-px^2+q).$$

Illustrative example IV:

$$\begin{aligned}x^2-6x+9-4y^2&=(x^2-6x+9)-4y^2\\&=(x-3)^2-4y^2\\&=(x-3+2y)(x-3-2y).\end{aligned}$$

Illustrative example V:

$$\begin{aligned}x^6-m^4-6m^2-9&=x^6-(m^4+6m^2+9)\\&=x^6-(m^2+3)^2\\&=(x^3+m^2+3)(x^3-m^2-3).\end{aligned}$$

Illustrative example VI:

$$\begin{aligned} a^2 + b^2 - 2mn^2 - n^4 + 2ab - m^2 &= (a^2+2ab+b^2) - (m^2+2mn^2+n^4) \\ &= (a+b)^2 - (m+n^2)^2 \\ &= (a+b+m+n^2)(a+b-m-n^2). \end{aligned}$$

Care must be used in the last type, because you sometimes will have to arrange the terms. They are not always arranged in the proper order in an example.

Exercises

Factor each of the following, and check as directed:

1. $x^2 - 81$.
2. $4m^2 - 121$.
3. $x^2 - 169a^2$.
4. $9a^2 - 25b^2$.
5. $36x^2 - 25m^4$.
6. $r^2s^4 - 144x^6$.
7. $225a^2b^2 - 121c^8$.
8. $9s^6 - 16t^{10}$.
9. $4(a+b)^2 - c^2$.
10. $x^2 - (y-2z)^2$.
11. $a^2 - 6ab + 9b^2 - c^2$.
12. $a^4 + 16b^4 - 4x^2 - 8a^2b^2$.
13. $x^2 - y^2 + 2yz - z^2$.
14. $4h^4 - 16m^2 - 8mn - n^2$.
15. $16h^2 - 25k^2 + 70kl - 49l^2$.
16. $z^4 - m^4 - s^4 - 2m^2s^2$.
17. $a^2 - 6ab + 9b^2 - 9x^2 + 6xy - y^2$.
18. $4p^2 + 9q^2 - 16s^2 - 25t^4 - 12pq + 40st^2$.
19. $a^{2n} - b^{6m}$.
20. $x^{4n+4} - y^{6m+8}$.
21. $.04x^2 - .09y^4$.
22. $1.44a^2 - 1.21b^6$.
23. $\frac{1}{9}x^4 - \frac{4}{25}y^2$.
24. $.0004a^2 - \frac{1}{16}b^2$.
25. $9x^2 - 4a^2 + 49y^2 + 42xy$.

37. Expressions Reducible to the Difference of Squares. — A trinomial of the form $x^4 + kx^2y^2 + y^4$ can often be reduced to the difference of squares by adding and subtracting the same quantity. For example, we shall consider $x^4 + x^2y^2 + y^4$. It will not change the value, but only the form if we add x^2y^2

to the middle term and subtract the same at the end. This reduces the expression to $x^4 + 2x^2y^2 + y^4 - x^2y^2$, which is evidently the difference of squares. You will notice that this expression may be reduced at once to the original one by merely collecting similar terms. Further steps in factoring this expression are

$$(x^4 + 2x^2y^2 + y^4) - x^2y^2 = (x^2 + y^2)^2 - x^2y^2$$
$$= (x^2+y^2+xy)(x^2+y^2-xy).$$

Rule. — (1) *Determine what perfect square may be added to the middle term of the trinomial to make it a perfect trinomial square.*

(2) *Add this quantity to the middle term and subtract it at the end. The result will be the difference of two squares.*

(3) *Factor by section 36.*

Illustrative example I:

Factor $a^4 + 2a^2b^2 + 9b^4$.

Solution: In order that this may be a perfect trinomial square it is necessary that the middle term shall be $\pm 6a^2b^2$. Hence we add and subtract $4a^2b^2$.

$$a^4 + 2a^2b^2 + 9b^4 = a^4 + 6a^2b^2 + 9b^4 - 4a^2b^2$$
$$= (a^2 + 3b^2)^2 - (2ab)^2$$
$$= (a^2 + 3b^2 + 2ab)(a^2 + 3b^2 - 2ab).$$

The above process is sometimes applicable to a binomial.

Illustrative example II:

Factor $x^8 + 64$.

Solution:

$$x^8 + 64 = x^8 + 16x^4 + 64 - 16x^4$$
$$= (x^4 + 8)^2 - (4x^2)^2$$
$$= (x^4 + 8 + 4x^2)(x^4 + 8 - 4x^2).$$

Why is it necessary that the given expression should be of a degree which is a multiple of 4, in order to factor by this method?

Exercises

Factor and check as directed:

1. $x^4 + 7x^2y^2 + 16y^4$.
2. $x^4 + 4x^2y^2 + 16y^4$.
3. $x^4 - x^2y^2 + 16y^4$.
4. $x^4 - 12x^2y^2 + 16y^4$.
5. $16m^4 - 17m^2n^4 + n^8$.
6. $81a^4 + 11a^2b^2 + 4b^4$.
7. $36a^4 - 25a^2b^2 + 4b^4$.
8. $9x^4 + 26x^2y^2 + 25y^4$.
9. $121a^4 - 119a^2b^2 + 25b^4$.
10. $100m^8 + 71m^4a^2 + 36a^4$.
11. $36y^4 - 33y^2z^2 + 16z^4$.
12. $4m^4 - 8m^2n^2 + 49n^4$.
13. $16p^4 - 56p^2q^2 + 25q^4$.
14. $a^4 + 64b^4$.
15. $x^8 + 4y^8$.
16. $49x^4 - 127x^2y^2 + 81y^4$.
17. $x^{4n} + x^{2n}y^{2m} + y^{4m}$.

Factor in two different ways:

18. $a^4 - 17a^2 + 16$.
19. $16x^4 - 104x^2y^2 + 25y^4$.

Exercises

Review Exercises in Special Products

Expand each of the following:

1. $(3a + 5)^2$.
2. $(4m + 2n^2)^2$.
3. $(5a - 2yz)^2$.
4. $(6a + b)(6a - b)$.
5. $(3a + 2b - c)(3a - 2b + c)$.
6. $2ab(3a - 2b + 2abc)$.
7. $(3a + 2b)(5a - 7b)$.
8. $(11a + 3)(2a - 1)$.
9. $(1 + 3a)(1 - 5a)$.
10. $(6a^2 + 7)(5a^2 - 1)$.
11. $(x + 2y - 1)(x + 2y + 1)$.
12. $(a - x + 4)^2$.
13. $(3a + m - 5 + 2b^2)^2$.
14. $(5x + 3)(2x - 7)$.
15. $(1 - 4xy)(3 + 7xy)$.
16. $(3a - 2b + 1)(3a + 2b - 1)$.
17. $(x - 6y + z)^2$.
18. $-3ab^2(2a^2 + 3ab - b^2)$.
19. 81^2.

Hint: $81^2 = (80 + 1)^2$.

20. 79^2.

21. 102^2.

22. 58^2.

23. $104 \cdot 96$.

24. $77 \cdot 83$.

25. $19 \cdot 21$.

26. $991 \cdot 1009$.

27. $(x^n + 3\,y^m)^2$.

28. $(.8\,x^2 + .5\,y^3)^2$.

29. $(x^a + 3\,x^b)(x^a - 4\,x^b)$.

30. $(3\,x - 2\,b + 4\,c)^2$.

31. $(a - b - 3\,c)^2$.

32. $(3\,x - 4\,y + 5\,z)^2$.

33. $(a^{2x} + a^x - 3)^2$.

34. $(2\,x + 3\,y)$ times $? = (2\,x^2 - xy - 6\,y^2)$.

35. How do your results compare in the following?

(a) $(-a - 7)^2$.

(b) $(a + 7)^2$.

36. $(5 - 9 + 6)^2 = ?$

Hint: First square as given, then simplify and square your result and compare it with the first result.

37. $16\,x^2 + (?) + 9$ will make this a perfect trinomial square?

38. $[(a + 2\,b) + c]^2$.

39. $[(a + 2\,b) + c][(a + 2\,b) - c]$.

40. $(.3\,x + .5\,y - .6\,z)^2$.

41. $(\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z)^2$.

Exercises

Review Exercises in Factoring

Factor each of the following, *being sure to obtain prime factors:*

1. $ax^2 - 36\,a$.

2. $2\,x^4 - 12\,x^2 + 18$.

3. $2\,a^2 - 11\,a + 21$.

4. $ax - 9\,x + ay - 9\,y$.

5. $x^4 - 81$.

6. $a^2x^2 - 9\,x^2$.

7. $2\,ax^2 + 18\,ax - 44\,a$.

8. $x^8 - 1$.

9. $2\,a^4b^4 - 128$.

10. $a^2 - b^2 + x^2 - 2\,ax$.

11. $a^2x^3y+4\,ax^2y^2-5\,a^3x^3y^2$.

12. $a^4 - a^2b^2 + 16\,b^4$.

13. $a^4yx^2 - a^4yx - 2\,a^4y$.

14. $12\,am+n+4\,m+3\,an$.

15. $a^2x^2 - 5\,ax - 6$.

16. $(a+b)^2 - 4(a+b)+4$.

17. $2 + 3\,a - 20\,a^2$.

18. $x^2 + \frac{1}{6}\,x - \frac{1}{6}$.

19. $4\,anx + 12\,nx + 2\,any + 6\,ny$.

20. $100\,n + 35\,an - 30\,a^2n$.

21. $400 - 9\,x^8$.

22. $x^4 - 10\,x^2 + 9$.

23. $x^5 - 3\,x^3 - 4\,x$.

24. $16\,x^2 - .0009$.

25. $H^2 - .64\,H + .024$.

26. $K^2 - .8\,K + .15$.

27. $2\,H-4\,Hx-30\,Hx^2$.

28. $x^2 - 16(a+b)^2$.

29. $x^2+4(a+b)x+4(a+b)^2$.

30. $x^4 - 8\,x^2 + 16$.

31. $16(x-y)^2-b^2a^2(x-y)^2$.

32. $\frac{1}{4}\,x(a+b)-\frac{1}{4}\,y(a+b)$.

33. $a^{4x} - b^{2y}$.

34. $a^{3x+2}+2\,a^{3x+1}b+a^{3x}b^2$.

35. $x^4 - x^2 + x + 1$.

36. $8\,x^4 + 40\,x^3 - 22\,x^2y^2$.

37. $9\,x^4 - 66\,x^2y^2 + 121\,y^4$.

38. $1 - x + x^4 - x^5$.

39. $x^{4n} - 256\,y^{8m}$.

40. $x^2 - 4 - 8\,x + 16$.

41. $x^7 - x^5 - x^2 + 1$.

42. $16\,x^4y^{10} - 160\,x^2y^6z^2 + 400\,y^2z^4$.

43. $8\,a^{2x} + 24\,a^x - 54$.

44. $x^{3a} + x^{2a} + x^a + 1$.

45. $a^2 + 16\,b^2 + 10\,xy + 8\,ab - x^2 - 25\,y^2$.

46. $9\,x^{2a} - 24\,x^a - 48$.

47. $.2\,x^4 + .4\,x^2 - .5\,x^5 - x^3$.

48. $1 - (3.5)^{4x}$.

49. $x^2 + \frac{7}{12}\,x + \frac{1}{24}$.

50. $19\,x^{15}+95\,x^{10}+17\,x^5+85$.

51. $12(a+1)^2-9(a+1)-30$.

52. $x^{2a+4} - x^2$.

53. $(x-y)(a^2-b^2)+(a+b)(x^2-y^2)$.

54. $x^2 - 16\,y^2 - x - 4\,y$.

55. $(x^2+x)^2+2(x^2+x)(x+1)+(x+1)^2$.

56. $(4\,x^2+9)^2 - 24\,x(4\,x^2+9)+144\,x^2$.

57. $x^{a+1} - x + x^{a-1}$.

58. $ax + bx + cx - ay - by - bz - cy - cz - az$.

59. $2\,x^{2a+2} - 3\,x^{a+1} - 2$.

60. $x^8 - 167\,x^4y^4 + y^8$.

38. Sum or Difference of Cubes. — The expression $a^3 + b^3$ may be divided by $a + b$ and the quotient is $a^2 - ab + b^2$.

Likewise $a^3 - b^3$ may be divided by $a - b$ and the quotient is $a^2 + ab + b^2$.

Hence $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

You will notice that the sign in the binomial factor agrees with the sign in the given binomial, while the sign of the second term of the trinomial factor is opposite to that of the second term of the binomial.

Illustrative example I:

Factor $a^3x^3 - 8$.

Solution: $a^3x^3 - 8 = (ax - 2)(a^2x^2 + 2\,ax + 4)$.

Illustrative example II:

Factor $x^6 + a^3$.

Solution: $x^6 + a^3 = (x^2)^3 + a^3$

$= (x^2 + a)(x^4 - ax^2 + a^2)$.

Exercises

Factor and check:

1. $a^3 - x^3$.
2. $x^3 - 8$.
3. $27 + y^3$.
4. $a^3x^3 + 27\,b^3$.
5. $64\,a^3 - 1$.
6. $(a + b)^3 - c^3$.
7. $x^3 + (a - b)^3$.
8. $8\,a^3 - 27\,c^3d^3$.
9. $x^3 + 8(c+d)^3$.
10. $x^6 + y^6$.
11. $a^3x^3 + 27\,y^6$.
12. $125\,n^6 - 8\,m^3$.
13. $x^3 - \frac{1}{y^3}$.
14. $x^3y^6z^9 - 27\,h^{15}$.
15. $x^{3a} - 1$.

16. $1 - a^6$.
17. $x^3 - y^3 - 2x^2y + 2xy^2$.
18. $125x^{6a} - 27y^{9b}$.
19. $x^3 - .027$.
20. $(a + b)^3 + (a - b)^3$.
21. $4 - x^2 + 4x^3 - x^5$.
22. $2a^2b^3 + 432a^2$.
23. $y^6 + x^9$.
24. $x^9 + y^9$.
25. $8x^{3n}y^m - 729y^{m+3}z^6$.

39. Sum or Difference of Odd Powers. — The sum of two like odd powers may be divided by the sum of their roots. The polynomial factor will have as many terms as the exponent of the like odd powers. For example:

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

Note the following facts concerning the polynomial factor:

(a) The number of terms is 5, as indicated by the exponent of the like odd powers.

(b) The first term (x^4) may be obtained by dividing x^5 by x.

(c) Powers of x decrease by 1 in each term after the first, disappearing in the last term.

(d) y appears in the second term in the first power, and its exponent increases by 1 in each subsequent term, reaching y^4, the quotient of y^5 by y, in the last term.

(e) The signs alternate.

In a similar manner

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

The rule for the sum and difference of cubes explained in the preceding section is a special case of the above rule.

Illustrative example I:

Factor $x^5 - 32$.

Solution:

$$\begin{aligned} x^5 - 32 &= x^5 - 2^5 \\ &= (x - 2)(x^4 + x^3 \cdot 2 + x^2 \cdot 2^2 + x \cdot 2^3 + 2^4) \\ &= (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16). \end{aligned}$$

Illustrative example II:

Factor $x^{10} + y^{15}$.

Solution:

$$x^{10} + y^{15} = (x^2)^5 + (y^3)^5$$

$$= (x^2 + y^3)[(x^2)^4 - (x^2)^3(y^3) + (x^2)^2(y^3)^2 - (x^2)(y^3)^3 + (y^3)^4]$$

$$= (x^2 + y^3)(x^8 - x^6y^3 + x^4y^6 - x^2y^9 + y^{12}).$$

Exercises

Factor:

1. $x^5 - y^5$.
2. $x^7 + a^7$.
3. $a^5b^5 + 32$.
4. $x^{10} - 32$.
5. $a^7 - 128$.
6. $n^5 + 243\,m^5$.
7. $x^7 + 128\,a^7$.
8. $(a + b)^5 - c^5$.
9. $m^5 + (c - d)^5$.
10. $x^5y^5 - 32$.
11. $a^{15} + 1$.
12. $32\,x^5y^{15n} - 243\,z^{10m}$.

40. The Factor Theorem. — A theorem is a statement of a mathematical truth. Theorems may be algebraic as well as geometrical. The factor theorem may be stated thus:

If a polynomial involving some letter such as x *may be reduced to 0 by substituting for* x *some quantity such as* a, *then* $x - a$ *is a factor of this polynomial.*

Proof:

Let P represent the polynomial.

Let a represent the quantity which reduces it to 0.

Divide P by $x - a$, and let Q represent the quotient and R represent the remainder.

Then $$P \equiv (x - a)Q + R.$$

In this identity, substitute a for x.

Then by hypothesis P becomes 0, and we obtain

$$0 \equiv 0 \cdot Q + R$$

or $$R = 0.$$

That is, the remainder is 0. Hence $x - a$ is an exact divisor or factor of P.

Discussion:

Why is the identity sign used in the first equation of the above proof?

Why have we a right to substitute a for x?

Illustrative example I:

Factor $x^3 - x^2 - x - 2$.

Solution: Substituting 2 for x, we obtain

$$8 - 4 - 2 - 2 = 0.$$

Therefore $x - 2$ is a factor. Dividing $x^3 - x^2 - x - 2$ by $x - 2$, we obtain as the other factor $x^2 + x + 1$.

Therefore $x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1)$.

Illustrative example II:

Factor $x^4 + 2x^3 - x^2 + 4x - 6$.

Solution: Evidently we need to try no numbers except such as are exact divisors of 6. The only possibilities then are $\pm 1, \pm 2, \pm 3, \pm 6$

Substituting $+1$ for x, we obtain

$$1 + 2 - 1 + 4 - 6 = 0.$$

Hence $x - 1$ is a factor.

Dividing the given polynomial by $x - 1$, we obtain

$$x^3 + 3x^2 + 2x + 6.$$

We try in succession $-1, 2, -2, 3, -3$, etc.

None of these will produce 0 until we come to -3.

Then we obtain

$$81 - 54 - 9 - 12 - 6 = 0.$$
$$81 - 81 = 0.$$

Therefore $x - (-3)$ or $x + 3$ is a factor.

Dividing $x^3 + 3x^2 + 2x + 6$ by $x + 3$, we obtain $x^2 + 2$.

Hence

$$x^4 + 2x^3 - x^2 + 4x - 6 = (x - 1)(x + 3)(x^2 + 2).$$

We now state the following

Rule. — (1) *Remove from the given expression all common monomial factors.*

(2) *Write down all possible exact divisors of the last or constant term.*

(3) *Substitute these in succession for* x *(or whatever letter is used in the polynomial) until one is found which makes the polynomial become 0.*

(4) *Write* $x -$ *this number as a factor.*

(5) *Remove this factor from the polynomial by division.*

(6) *So continue until the resulting quotient is of the 2d degree or is prime.*

(7) *Factor this resulting quotient by inspection if possible.*

(8) *Write the original polynomial as the product of all factors found.*

Exercises

Factor each of the following:

1. $x^3 - 3x^2 + 2x - 6$.
2. $x^3 - x^2 + 4x - 4$.
3. $x^3 - 2x^2 - 9$.
4. $x^3 + 2x^2 + 3x - 6$.
5. $x^3 - 7x + 6$.
6. $x^3 - 4x^2 + 4x - 16$.
7. $x^4 - 3x^3 - 6x^2 + 6x + 8$.
8. $x^4 - 5x^3 + 7x^2 - 5x + 6$.
9. $2x^4 + x^3 - 14x^2 - 19x - 6$.
10. $8x^4 - 34x^3 + 43x^2 - 11x - 6$.
11. $3x^2 + 4x - 15$.
12. $x^4 - 2x^2 + 1$.
13. $y^4 - 7y + 6$.
14. $y^3 - 27y - 10$.

Show without actual division

15. That $x^4 - 16$ is divisible by $x - 2$ and by $x + 2$.

16. That $x^5 - 1$ is divisible by $x - 1$, but not by $x + 1$.

17. That $x^3 - 2x^2y + 2xy^2 - y^3$ is divisible by $x - y$.

18. That $x^3 - 3x^2 + 3x - 1$ is divisible by $x - 1$.

19. Find all factors of example 18 by the above method.

20. Factor $x^3 - 1 - 3x^2 + 3x$ by grouping.

How do your factors compare with those of example 19?

21. Is $x^n - y^n$ exactly divisible by $x - y$?

41. Summary of Cases of Factoring.

1. $abc^2 + abc + ab = ab(c^2 + c + 1)$.
2. $ax + bx + ay + by = (a + b)(x + y)$.
3. $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$.
4. $a^2 + 2ab + b^2 = (a + b)^2$.
5. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = (a + b + c)^2$.
6. $x^2 - y^2 = (x + y)(x - y)$.
   $x^2 - (a + b)^2 = (x + a + b)(x - a - b)$.
   $(x + y)^2 - (a + b)^2 = (x + y + a + b)(x + y - a - b)$.
7. $x^4 \pm kx^2y^2 + y^4 = (x^2 + y^2 + lx)(x^2 + y^2 - lx)$.
8. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
9. $a^5 \pm b^5 = (a \pm b)(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4)$.
10. The factor theorem.

42. General Suggestions concerning Factoring.

1. Always examine the given expression first to determine the possibility of removing from each term a common monomial factor.

2. After having resolved the expression into two factors be sure to examine each to determine whether further factoring is possible. *Factors must be prime.*

3. The difference of even powers should always be factored as the difference of squares.

Thus

$$x^6 - y^6 = (x^3)^2 - (y^3)^2$$
$$= (x^3 + y^3)(x^3 - y^3)$$
$$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

4. The sum or difference of odd powers must be made the sum or difference of *like* odd powers.

Thus $x^5 + y^{15} = x^5 + (y^3)^5$, etc.

5. The sum or difference of odd powers should be considered as the sum or difference of the *lowest* possible odd powers.

Thus $a^9 - b^9 = (a^3)^3 - (b^3)^3$
$= (a^3 - b^3)(a^6 + a^3b^3 + b^6)$
$= (a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6).$

6. The sum of even powers cannot be factored unless the exponents of the even powers have a like *odd* factor.

Thus $x^4 + y^4$ is prime, but $x^6 + y^6$ can be factored for

$$x^6 + y^6 = (x^2)^3 + (y^2)^3$$
$$= (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

Likewise

$$x^{10} + y^{10} = (x^2)^5 + (y^2)^5, \text{ etc.}$$

7. It is often helpful to substitute temporarily a single letter for a parenthesis.

Thus, factor $6(m + n)^2 + 7\,y(m + n) - 5\,y^2$.

Substitute x for $m + n$. Then we have

$$6\,x^2 + 7\,xy - 5\,y^2.$$

The factors of this expression are

$$(3\,x + 5\,y)(2\,x - y).$$

Now substitute $m + n$ for x, and we obtain

$$(3\,m + 3\,n + 5\,y)(2\,m + 2\,n - y).$$

8. Terms may be rearranged, but *no signs may be changed.*

Exercises

43. General Review Exercises in Factoring.

Factor each of the following, and check as directed:

1. $x^3 - x$.
2. $x^6 - 1$.
3. $x^8 + x^2$.
4. $2x^3 + x^2 - 21x$.
5. $x^{12} + 1$.
6. $x^{12} - 1$.
7. $lx + 2cx - 6cy - 3ly$.
8. $a^2 + 9c^2 - d^2 - 6ac$.
9. $x^4 - 10x^2 + 9$.
10. $x^3 + 5x^2 + 2x - 8$.
11. $8x^3 - 125$.
12. $64 + 27a^9$.
13. $x^4 - a^2x^2 + 16a^4$.
14. $18ax + 9a^2x - 189a^3x$.
15. $225 - x^6$.
16. $x^4 + 4y^4$.
17. $4 - 11a - 3a^2$.
18. $x^9 - a^{15}$.
19. $3x^6 - 2x^3a^3 - a^6$.
20. $x^{2n} - 6x^n + 9$.
21. $x^{3n} - y^{3m}$.
22. $x^{2a+1} + x^{a+1} - 6x$.
23. $24xy - 8y + 18x - 6$.
24. $lm^2 - km^2 - ln^2 + kn^2$.
25. $a^2 - 2ab - 9x^2 - y^2 + b^2 + 6xy$.
26. $4(x - y)^3 - (x - y)$.
27. $(1 + x^3) + (1 + x)^3$.
28. $x^4 - 3x^2 - 4$.
29. $a^4 - 4a^3 - a^2 + 16a - 12$.
30. $2y + x^2 - x - 4y^2$.
31. $\frac{a^3}{8} - 27b^6$.
32. $x^{3m} - x^{2m} + x^{2m+1} - x^{m+1}$.
33. $(a^2 - b^2)^2 - (a^2 - ab)^2$.
34. $x^4 - 8x - x^3 + 8$.
35. $18x^2y + 60xy^2 + 50y^3$.
36. $24x^2y^2 - 47xy - 75$.

37. $(2x^2 + 3y^2)h + (2h^2 + 3x^2)y$.

38. $(x - x^2)^3 + (x^2 - 1)^3 + (1 - x)^3$.

39. $x^2 - (a + 5)x + 5a$.

40. $x^3 + 5x^2 - 29x - 105$.

41. $x^4 + 4$. *Hint:* Add and subtract $4x^2$.

42. $\dfrac{x^2}{y^2} - 9$.

43. $13x^4 + 104x$.

44. $6^{3a} - 1$. Two factors only.

45. $3^{4a} - 9^{2b}$. Two factors only.

46. $x + 5 + (x + 5)^2$.

47. $x^2 - 6x - 216$.

48. $3^{3a} - 2^{3b}$. Two factors only.

49. $x^3 - 4x^2 + 8$.

50. $32x^6 + x$.

51. $1 + x^5$.

52. $2x^6y - 64xy^6$.

53. $1 - (x - y)^3$.

54. $x^{3a} - 27$.

55. $27x^5 - 8x^2$.

56. Is $x - 3$ a factor of $x^3 + 3x^2 - 15x - 9$?

57. Is $x + 1$ a factor of $x^4 + 3x^3 - 4x^2 + x + 7$?

58. Is $x + 2$ a factor of $x^5 + 4x^2 + 16$?

Factor:

59. $2x^3 + 7x^2 + 4x + 3$.

60. $x^3 + 5x^2 + 7x + 3$.

61. $4(a - b)^2 - 12(a - b)(x + y) + 9(x + y)^2$.

62. $15(a + b)^2 - 38(a + b)(x + y) + 24(x + y)^2$.

63. $(x^2 + 5y - 3)^2 - 31(x^2 + 5y - 3) + 150$.

64. $x^3 - y^3 + 3y^2z - 3yz^2 + z^3$.

65. $.16x^2 - .04y^2$.

66. $.09x^2 - .24xy + .16y^2$.

67. $\frac{1}{16}x^2 - \frac{1}{9}y^2$.

68. $\frac{1}{9}x^2 + \frac{1}{3}xy + \frac{1}{4}y^2$.

69. $x^4y^{a+1} + x^2y^{a+3} + y^{a+5}$.

70. $\pi x^3 + \pi$.

71. $Rx^3 - 3\,Rx^2 - 4\,Rx + 12\,R.$

72. $(2\,y^2 - 3)^2 - y^2.$

73. $(x^3 - 8\,y^3) - h(x - 2\,y)^2.$

74. $x^8 - x^5 + 32\,x^2 - 32.$

75. $x^3 + 2\,x + 1 + 2\,x^2.$

76. $x^{4a} + x^{3a} + x^{2a} + x^a.$

77. $x^4 - 7\,x^2 + 1.$

78. $(x^3 + 3\,x)^2 - (3\,x^2 + 1)^2.$

79. $x^2 - \frac{3}{4}\,xy - \frac{1}{4}\,y^2.$

80. $x^8 + x^4 + \frac{1}{4}.$

81. $\dfrac{a^2}{b^2} + \dfrac{2\,a}{b} - 3.$

82. $x^{a+1} + x^a y + xy^a + y^{a+1}.$

83. $21\,x^2 - 23\,x + 6.$

84. $4\,x^4 + y^4 - 5\,x^2y^2.$

85. $a^3x - ab^2x - a^2c + b^2c + a^2by - b^3y.$

86. $18\,x^2y + 60\,xy^2 + 50\,y^3.$

87. $3^{3a} - 5^{3b}.$

88. $a^2 - 1 + 2\,x - x^2.$

89. $5\,x^3 - 26\,ax^2 + 5\,a^2x.$

90. $(x - a)^4 - 16\,a^4.$

91. $x^3 - 4\,x^2 + 5\,x - 2.$

92. $x^{n+2} - ax^{n+1} + bx^{n+1} - abx^n.$

93. $x^3 + x + y^3 + y.$

94. $3\,x^2 + 19\,x - 14.$

95. $3\,x^2 + 2\,xy - 40\,y^2.$

96. $(x^2 - y^2)(x + y) + 2\,xy^2 - 2\,x^2y.$

97. $(x + y)^2 - 23(x + y) + 76.$

98. $\frac{9}{16}\,a^2(c^2 - d^2) - \frac{9}{25}\,b^2(c^2 - d^2).$

99. $x^3 - 15\,x^2 + 72\,x - 110.$

100. $2(a + b)^2 - 4.3(a + b)(x + y) + .21(x + y)^2.$

44. Highest Common Factor. — A factor which occurs in each of two or more expressions is a common factor of these expressions. The product of all prime common factors of two or more expressions is their *Highest Common Factor.*

Rule. — *To find the highest common factor (H. C. F.) of two or more expressions:*

(1) *Resolve each expression into its prime factors.*

(2) *Indicate the product of all factors common to all the expressions.*

(3) *The result is their highest common factor.*

The *highest common factor* of arithmetical numbers is often called their *greatest common divisor.*

Illustrative example I:

Find the H. C. F. of $24\,ax^3y^2$, $30\,a^2xy^4$, and $42\,ax^2y^5$.

Solution: $24\,ax^3y^2 = 2 \cdot 2 \cdot 2 \cdot 3\,ax^3y^2.$

$30\,a^2xy^4 = 2 \cdot 3 \cdot 5\,a^2xy^4.$

$42\,ax^2y^5 = 2 \cdot 3 \cdot 7\,ax^2y^5.$

$\therefore$ H. C. F. $= 2 \cdot 3 \cdot a \cdot x \cdot y^2 = 6\,axy^2.$

Illustrative example II:

Find H. C. F. of $2x^4 + 2x^3 - 2x^2 - 2x$, $4x^7 - 4x$, and $4x^5 - 4x$.

Solution:

$2x^4+2x^3-2x^2-2x = 2x(x+1)^2(x-1).$

$4x^7 - 4x = 4x(x+1)(x-1)(x^2-x+1)(x^2+x+1).$

$4x^5 - 4x = 4x(x+1)(x-1)(x^2-1).$

$\therefore$ H. C. F. $= 2x(x+1)(x-1).$

Exercises

Find the H. C. F. of the following:

1. $16\,m^2n^3p^4$, $40\,mn^5p^3$, $32\,m^3n^2p^5$, $72\,m^2n^6p^4$.
2. $22\,x^3(p-q)^2$, $33\,x^5(p-q)(p-r)$, $44\,x^4(p-q)^2(x-y)$.
3. x^2-1, x^2-2x+1, x^3-1, x^2+2x-3.
4. x^3-x, $x^6-x^4+x^3-x$, $(x^2+x)(x-1)^2$, x^7-x.
5. x^4+8x, $2x^3-2x^2-12x$, $4x^4+8x^3$.
6. x^2-y^2, $x^2-6xy+5y^2$.
7. x^2-4x-5, x^3-4x^2-4x-5, $x^2-5x+xy-5y$.
8. $1-x^2$, $15-16x+x^2$, $1-x^3$.

9. $(x^2 - 1)(x^2 + 5x + 6)$, $(x^2 + 3x)(x^2 - x - 6)$.

10. $(xy - y^2)^3$, $y^3 - x^2y$, $x^3 - y^3$, $y^2 - 2xy + x^2$.

11. 225, 325, 475, 550.

12. 2^5, 2^7, 2^3, 2^9, 2^4.

13. $x^2 - b^{2a}$, $x^2 - 3xb^a - 4b^{2a}$, $x^3 + b^{3a}$.

14. $x^2 - 5x - 36$, $x^2 - 16$.

15. $x^2 - (y + z)^2$, $(y - x)^2 - z^2$, $y^2 - (x - z)^2$.

16. $(x^2y - xy^2)^2$, $xy(x^2 - y^2)$.

17. $x^2 - 3x - 40$, $x^2 - x - 30$, $x^2 + 3x - 10$.

18. $x^{2a} - y^{2a}$, $x^{2a} - 2x^ay^a + y^{2a}$, $x^{2a} - x^ay^a$.

19. $4x^3 + x - 5$, $6x^3 - 5x^2 - 1$.

20. $x^5 - 11x^4 + 42x^3 - 71x^2 + 61x - 30$ and $x^3 - 9x^2 + 26x - 24$.

21. Write two expressions whose H. C. F. is $(x + 1)$.

22. Write two expressions whose H. C. F. is $(x - 7)$.

23. Write three expressions whose H. C. F. is $.5x(x - a)$.

24. Write the H. C. F. of $x^2 - 9$, $x^2 - x - 6$, and $(x - 3)^2$.

45. Lowest Common Multiple. — An algebraic expression into which each of several other expressions can be divided without a remainder is a common multiple of these expressions. The *lowest common multiple* (L. C. M.) of several expressions is that number *of lowest degree* (or, if numerical, the *least number*) into which each of them can be exactly divided.

For example, $48a^3b^3$, $96a^5b^8$, $144a^3b^{10}$, $576a^5b^4$, etc., are common multiples of $6a$, $8a^2$, and $12ab$, but the *lowest* common multiple of these quantities is $24a^2b$.

Rule. — *To find the lowest common multiple of two or more expressions:*

(1) *Resolve each of these expressions into its prime factors.*

(2) *Write the indicated product of every different factor found, taking each factor the greatest number of times it occurs in any one expression.*

Illustrative example I:

Find the lowest common multiple of

$24\,x^2(x-y)$, $36\,x^3(x-y)^2$, and $72\,x(x-y)(x+y)$.

Solution: 72 will contain both 24 and 36.

x^3 will contain both x^2 and x.

$\therefore$ L. C. M. $= 72\,x^3(x-y)^2(x+y)$.

Illustrative example II:

Find the L. C. M. of $2\,x^2-2$, $6\,x^3-6$, $x^2+2\,x-3$.

Solution:

$2\,x^2-2 = 2(x+1)(x-1)$.

$6\,x^3-6 = 6(x-1)(x^2+x+1)$.

$x^2+2\,x-3 = (x-1)(x+3)$.

$\therefore$ L. C. M. $= 6(x+1)(x-1)(x+3)(x^2+x+1)$.

Exercises

Find the L. C. M. of each of the following sets of expressions:

1. $5\,x^2y^2$, $20\,x^3y$, $25\,x^2y^3z$, $30\,xyz^4$.
2. $4\,xy(x-y)$, $6\,x^2y(x-y)(x+y)$, $8\,xy^3z(x+y)^2$.
3. $3\,x(x-y)(x^2-y^2)$, $12\,y(x-y)^3(x+y)$, $16\,xy(x^2-y^2)^2$.
4. x^2-x-2, $x^2-8\,x+12$, $x^2-5\,x-6$.
5. x^3-8, $x^3-6\,x^2+12\,x-8$, $5\,x^2+10\,x+20$.
6. x^2-x-6, $x^2-6\,x+9$, $6\,x-18$.
7. ax^2-a, $2\,ax+2\,a$, x^2+x-2.
8. a^2-b^2, $a^2-2\,ab-3\,b^2$, $(a+b)^2$.
9. 165, 220, and 275.
10. x^3-8, $4-x^2$, $x^3+4\,xy^2+2\,x^2y$.
11. $x^{2a}-x^a-2$, $x^{2a}-4$, $x^{3a}-8$.

12. $x^2 + 2x - 1,\ 2x^3 + x^2 - 8x + 3.$

13. $x^2 + 4x + 4,\ x^2 + 5x + 6,\ x^2 + 3x + 2.$

14. $x^2 + xy,\ xy + y^2,\ x^2 - xy - 6y^2.$

15. $x - y,\ y - x.$

16. $12(x - y)^2,\ 8(y - x)^2.$

17. $x^5 - 16x,\ x^5 - 3x^3 - 4x,\ x^5 + 5x^3 + 4x.$

18. $9x^2 - x^4,\ x^2 - 5x + 6,\ 10 - x - 2x^2.$

19. $6x^2 + 18x - 60,\ 3x^2 + 24x + 45,\ 8x^2 - 24x + 16.$

20. $x^2 + 4yx - 5y^2,\ x^2 + 2yx - 15y^2,\ x^2 - 4yx + 3y^2.$

21. $a^2 - b^2,\ (a - b)^2,\ a^3 - b^3.$

22. $x^{2a} - 5x^a - 6,\ x^{2a} - 1,\ x^{2a} + 4x^a + 3.$

23. $(x^2 - 1)(x^2 + 5x + 6),\ (x^2 + 3x)(x^2 - x - 6).$

24. $x^2 + x - 20,\ x^4 - 3x^3 - 2x^2 - 8x.$

25. $.15x^2 + 2.06x + .8,\ .25x^2 - .04.$

46. Simple Equations Solved by Factoring. — The way in which factoring may be used as an aid in solving simple literal equations may be made clear by the following:

Illustrative example I:

Solve for x, $hx - h^3 = -k^3 + kx$.

Solution:

Transposing, $hx - kx = h^3 - k^3.$

Factoring, $x(h - k) = (h - k)(h^2 + hk + k^2).$

By axiom of division, $x = h^2 + hk + k^2$, root.

Illustrative example II:

Solve $nx + 2n - mn = 2ml - 4l - 2lx$.

Solution:

Transposing, $nx + 2lx = mn - 2n + 2ml - 4l.$

Factoring, $x(n + 2l) = (n + 2l)(m - 2).$

By axiom of division, $x = m - 2$, root.

Exercises

Solve each of the following equations for x:

1. $ax + 3x = a^2 - 9.$
2. $mx - 2nx = m^2 - 5mn + 6n^2.$
3. $lx = 3l^2 - 2ls - sx - 5s^2.$
4. $bx + cx = b^2 + 3bc + 2c^2.$
5. $3bx + 4ax = 9b^2 + 24ba + 16a^2.$
6. $b + 42 - b^2 = 7x - bx.$
7. $bx - b^2 + c^2 = cx.$
8. $5x - k^2 = 10k - kx + 25.$
9. $2x = 10 - 3a + ax - a^2.$
10. $2bx - sb = 3s - 6x.$
11. $ax - \frac{1}{2}x - \frac{1}{6} = a^2 - \frac{5}{6}a.$
12. $bx + cx = d.$
13. $(x - a)(x + b) = x^2.$
14. $ax - bx = a^3 - b^3.$
15. Is $x = a - b$ a root of $a^2x + 2abx + b^2x = (a + b)^2(a - b)$? Prove.
16. $.5ax - 4a = .5bx - 4b.$
17. $.2ax + 3a + 3b + .2bx = a + b.$

47. The Quadratic Equation Solved by Factoring. — A quadratic equation is an equation which involves the second power of the unknown quantity, but no higher power; as, for example, $2x^2 - x = 3$. Principles of factoring provide an easy means of solving this equation. Thus:

Transpose all terms to the left member, $2x^2 - x - 3 = 0.$

Factoring the left member, $(2x - 3)(x + 1) = 0.$

Set each factor equal to 0, $2x - 3 = 0$. $x + 1 = 0$.

$2x = 3$. $x = -1$.

$x = \frac{3}{2}$.

Therefore $\frac{3}{2}$ and -1 are the roots of this equation.

Check with $\frac{3}{2}$:

$2(\frac{9}{4}) - \frac{3}{2} - 3 = 0.$

$\frac{9}{2} - \frac{3}{2} - 3 = 0.$

$3 - 3 = 0.$

$3 = 3.$

Check with -1:

$2 + 1 - 3 = 0.$

$3 - 3 = 0.$

$0 = 0.$

The reason for setting each member equal to zero becomes clear when we consider that we have in this equation a product equal to 0. Now a product is equal to 0 if, and only if, one of its factors equals 0, and it makes no difference which one. It is sufficient, therefore, to satisfy the condition expressed in the equation, that either factor shall be 0.

Since not all quadratic trinomials can be factored, it is necessary to have other methods of solution in certain cases, and these will be discussed later.

From this point in our work occasional quadratic equations which can be solved by factoring will occur.

The following is a statement of the

Rule. — (1) *Transpose every term to the left member of the equation, thus making the right member 0.*

(2) *Factor the left member of the equation.*

(3) *Set each factor which contains the unknown quantity equal to 0, and*

(4) *Solve the resulting equations.*

(5) *Check your results.*

Equations of higher degree than the second can be solved by the above rule, provided only that the left member can be factored into linear, or first degree factors, after the right member has been made 0.

Exercises

Solve the following equations for the value of x, and check your roots obtained:

1. $x^2 - 4x + 3 = 0$.
2. $x^2 - 4x = 12$.
3. $3x^2 - 13x = -4$.
4. $2x^2 - x - 6 = 0$.
5. $4x^2 - 19x = 5$.
6. $6x^2 - 13x = -6$.
7. $10x^2 - 11x + 3 = 0$.
8. $4x^2 - 7x - 2 = 0$.
9. $2x^2 - 11x = -12$.
10. $3x^2 + 14x = 5$.
11. $2x^2 + 13x + 20 = 0$.
12. $6x^2 - 23x + 7 = 0$.
13. $7x^2 - 19x = 6$.
14. $5x^2 - 4x = 1$.
15. $21x^2 - 40x = 21$.
16. $2x^2 - 3x - 35 = 0$.
17. $33x^2 - 4x = 12$.
18. $9x^2 - 29x = 28$.
19. $6x^2 - 7x = 24$.
20. $4x^2 - 17x + 4 = 0$.
21. $24x^2 - 49x + 15 = 0$.
22. $15x^2 - 16x = 15$.
23. $2x^2 + 21x - 11 = 0$.
24. $2x^2 - 23x + 30 = 0$.
25. $x^2 + 4bx = 12b^2$.
26. $5x^2 + 4ax = a^2$.
27. $2x^2 = ax + 10a^2$.
28. $x^2 + 10x = 24$.
29. $2x^2 - 5x - 7 = 0$.

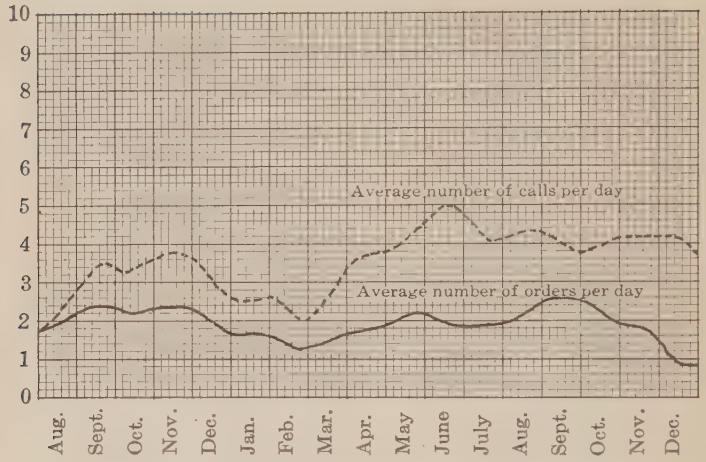
Solve the following equations of the 3d degree by factoring, and check as directed:

30. $x^3 - 9x = 0$.
31. $x^3 + x^2 = 0$.
32. $x^3 - x^2 - x + 1 = 0$.
33. $x^3 - 2x^2 - 5x + 6 = 0$.
34. $x^3 + 7x^2 + 12x = 0$.

Solve and check the following 4th degree equations:

35. $x^4 - 10x^2 + 9 = 0$.
36. $x^4 - 5x^2 + 4 = 0$.
37. $x^4 - 13x^2 + 36 = 0$.
38. $x^4 - 26x^2 = -25$.

Exercises



1. The adjoining chart is intended to show the salesman's *efficiency*, not so much in money, as his activity. That is, the number of calls which he makes in a day as compared with the number of orders taken per day. If the two curves continue approximately parallel, his efficiency is considered good, whereas a great difference in the directions of the two lines will show that his case is to be studied for some trend. It may be that he has gone "stale" and such a chart will show this fact.

Construct a chart somewhat similar to the above to show the following facts:

2.

	Jan.	Feb.	Mar.	Apr.	May	June
Average number of orders per day	3	5	6	4	3	5
Average number of calls per day	4	7	8	5	4	7

	July	Aug.	Sept.	Oct.	Nov.	Dec.
Average number of orders per day	7	6	7	6	8	7
Average number of calls per day	9	10	9	10	11	10

5. $\dfrac{x^3y^2 - xy^3 + x^4y}{xy}$.

6. $\dfrac{12\,x^3 - 6\,x^2 - 18\,x^5}{-\,6\,x}$.

7. $\dfrac{42\,a^5b^6c^4d^3}{-\,7a^2bc^4d^2}$.

8. $\dfrac{-\,120\,x^4y^3z^7 + 40\,x^2y^5z^4 - 30\,x^6y^2z^8}{-\,10\,x^2y^2z^4}$.

9. $\dfrac{-\,16\,a^{2m} - 6\,a^{6m}}{-\,2\,a^m}$.

10. $\dfrac{10x^{n+3} - 8x^{n+2} - 6x^{n+1}}{-\,2\,x}$.

11. $\dfrac{2^6}{2^3}$.

12. $\dfrac{-\,2^{10}}{2^5}$.

13. $\dfrac{-\,3^6}{-\,3^2}$.

14. $\dfrac{-\,2^8}{(-\,2)^8}$.

15. $\dfrac{3^2 \cdot 2^6}{-\,3 \cdot 2^4}$.

16. $\dfrac{-\,3^{n+1} \cdot 5^{n+6}}{3^n \cdot 5^{n+3}}$.

17. $(\frac{1}{4}\,x^2 + \frac{1}{8}\,x^3 + \frac{4}{7}\,x^6) \div \frac{1}{2}\,x^2$.

18. $.6\,x^2)1.2\,x^2 + 3\,x^3 - 6\,x^4$.

19. $\dfrac{8(x-y) + 4(x-y) - 6(x-y)}{2(x-y)}$.

20. What must $-\,H^2K^2$ be multiplied by to produce $H^3K^4 - 6\,H^2K^3 + 5\,H^4K^5$?

21. Divide: $\dfrac{.25}{-.5}$, $\dfrac{2.75}{-.25}$, $\dfrac{125}{.25}$, $\dfrac{7.75\,x^2}{-.25\,x}$.

Divide and check:

22. $18\,x^2y$ by $9\,x^2y$.

23. $.27\,x^2y^3$ by $.03\,x^2y^3$.

24. $\frac{1}{5}\,x^4y^7$ by $\frac{1}{3}\,x^4y^7$.

25. $16\,x^5$ by $4\,x^5$ by $\frac{1}{2}$.

26. $.6\,x^3y$ by $.3\,xy^2$.

27. $3\,x^4y^{5a+b}$ by xy^{3a+b}.

28. $6\,x^{2a+b+c}$ by $3\,x^{a+2b+3c}$.

29. $2.4(x+y)^3$ by $.6(x+y)$.

30. 2^{5x} by 2^{3x}.

31. $\dfrac{-\,36(x+y)^{4a+6}}{4(x+y)^{3a+3}} = ?$

Review

1. Define and illustrate:

(a) Negative number. (d) Absolute value.
(b) Coefficient. (e) An equation.
(c) Exponent.

2. If $x = 6$, write the right-hand member of the equations:

(a) $.5\,x + .75\,x =$ (c) $x^2 + 4\,x + .5\,x =$
(b) $\frac{1}{3}\,x + \frac{1}{2}\,x + \frac{1}{12}\,x =$

3. If $a = 4$, $b = 2$, $c = 0$, $d = 3$, find the value of

(a) $4\,ad^2$. (e) $\dfrac{3\,a - 2\,b}{d}$. (g) $d^3 + \dfrac{c}{b}$.
(b) $2\,a^3b$.
(c) $2\,a^2b - 7$. (f) $\dfrac{3\,a^2 + 4\,d^3}{5}$. (h) $\frac{1}{2}\,a^2b + \frac{1}{4}\,ab^2$.
(d) $3\,b + 4\,a$.

4. If my debts are diminished, am I richer or poorer? Prove it.

5. Express in algebraic symbols:

(a) Five times x plus six times y.

(b) Five times the cube of x minus four times the square of y.

(c) Four x square plus five times the quantity x minus y.

(d) Six x cube diminished by twice y square.

(e) One fifth x cube plus one fourth y cube minus one sixth z square.

6. Multiply $a^4 + 3\,ab^3 + b^4 - a^3b$ by $b^2 - a^2$.

7. Add, $3x^{2a+3} + 5\,x^{a+2} - 6\,x^{3a-1}$; $5x^{3a-1} - 4\,x^{2a+3} - 6\,x^{a+2}$; $9x^{a+2} - 7\,x^{2a+3} - 16\,x^{3a-1}$; $12x^{3a-1} + 2\,x^{2a+3} - 8\,x^{a+2}$.

8. Is 2, 6, or 3 a root of $.7\,x - .9\,x + .35\,x = .45$? Prove.

16. Division by a Polynomial Divisor. — The method of dividing one polynomial by another is indicated by the following

Illustrative example I:

Divide $2x^3 - 3x^2 + 7x - 3$ by $2x - 1$.

Solution:

$$\begin{array}{r|l} 2x^3 - 3x^2 + 7x - 3 & 2x - 1 \\ \cline{2-2} 2x^3 - x^2 & x^2 - x + 3 \\ \cline{1-1} -2x^2 + 7x & \\ -2x^2 + x & \\ \cline{1-1} 6x - 3 & \\ 6x - 3 & \\ \cline{1-1} \end{array}$$

The above example is worked by the aid of the following

Rule. — (1) *Arrange both dividend and divisor according to either the ascending or descending powers of a common letter.*

(2) *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

(3) *Multiply the entire divisor by the term of the quotient just found and subtract the result from the dividend.*

(4) *Divide the first term of the remainder by the first term of the divisor to obtain the second term of the quotient.*

(5) *Multiply the entire divisor by the second term of the quotient, subtract, and so continue until the divisor is exact or the remainder is of lower degree than the divisor.*

Care should be taken to keep the remainder arranged in the same order as the dividend and divisor.

For synthetic division see page 393.

Illustrative example II:

Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

Solution:

$$\begin{array}{r|l} x^4 + x^2y^2 + y^4 & x^2 + xy + y^2 \\ \cline{2-2} x^4 + x^3y + x^2y^2 & x^2 - xy + y^2 \\ \cline{1-1} -x^3y + y^4 & \\ -x^3y - x^2y^2 - xy^3 & \\ \cline{1-1} x^2y^2 + xy^3 + y^4 & \\ x^2y^2 + xy^3 + y^4 & \\ \cline{1-1} \end{array}$$

If there is a remainder, it should be written over the divisor in the form of a fraction, and the resulting fraction connected to the quotient by a + sign.

Illustrative example III:

Divide $x^4 + y^4$ by $x - y$.

Solution:

$$
\begin{array}{l|l}
x^4 + y^4 & x - y \\
\hline
x^4 - x^3y & x^3 + x^2y + xy^2 + y^3 + \dfrac{2y^4}{x - y} \\
\hline
x^3y + y^4 & \\
x^3y - x^2y^2 & \\
\hline
x^2y^2 + y^4 & \\
x^2y^2 - xy^3 & \\
\hline
xy^3 + y^4 & \\
xy^3 - y^4 & \\
\hline
+ 2y^4 &
\end{array}
$$

17. Checks in Division. — One of the best checks in division is multiplication. The dividend is equal to the product of the quotient and divisor, plus the remainder if any. If the division is exact, a numerical check is often advantageous. Thus in Illustrative example I above if $x = 2$, the dividend equals 15, the divisor 3, and the quotient 5. $15 \div 3 = 5$.

Exercises

Divide:

1. $x^2 - 3x - 10$ by $x - 5$.
2. $6a^2 + 5a - 21$ by $2a - 3$.
3. $3a^3 + 6a^2 - 5a - 10$ by $a + 2$.
4. $6n^3 - 3n^2 - 38n + 20$ by $2n - 5$.
5. $3x^3 - 22x^2 + 16x - 3$ by $x^2 - 7x + 3$.
6. $2x^4 + x^3 - 19x^2 - x - 28$ by $2x + 7$.

7. $3x^4 - 23x^3 - 7x^2 + 42x - 10$ by $3x^2 + x - 5$.

8. $2x^5 - 9x^4 + 15x^3 - 27x^2 + 20x - 4$ by $2x^2 - 7x + 2$.

9. $x^3 - 7x^2y + 3xy^2 - 21y^3$ by $x^2 + 3y^2$.

10. $2x^4 - 5x^3y + 4x^2y^2 - 2xy^3 - y^4$ by $x^2 - xy + y^2$.

11. $x^4 - y^4$ by $x - y$.

12. What must $x + y$ be multiplied by to produce $x^4 - y^4$?

13. What must $x - 2$ be multiplied by to produce $x^5 - 32$?

14. What must $x^4 + x^2 + 1$ be multiplied by to produce $x^8 + x^4 + 1$?

15. If the product is $m^6 + n^6$, the multiplier $m^2 + n^2$, what is the multiplicand?

16. $(x^2 - 2y^2 + xy - 3z^2 + 7yz + 2xz) \div (3z + x - y) = ?$

Divide:

17. $3x^2 - 2x^4 + x^5 - x^3 - 2x - 15$ by $x^3 - 5 - 4x$.

18. $x^6 - y^6$ by $x^2 + xy + y^2$.

19. $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

20. By what must the quantity $(x^2 - 2x + 1)$ be multiplied to produce $x^6 - 2x^3 + 1$?

Divide:

21. $x^5 + y^5 + x^4 + y^4 - x^4y - xy^4 - 2x^2y^2$ by $x^3 + y^3 - x^2y - xy^2$.

22. $x^3 + y^3$ by $x - y$.

23. $8y^3 - y^2 - 5$ by $2y - 3$.

24. $9x^4 - 67x^2 + 50x + 36$ by $x^2 + x - 6$.

25. $x^5 - y^5$ by $x - y$.

26. $6x^{3a} - 25x^{2a} + 27x^a - 5$ by $2x^a - 5$.

27. $x^{3a-1} - x^{2a} + 3x^{a-1} - 3$ by $x^{a-1} - 1$.

28. $6x^{2a} - 7x^ay^b - 20y^{2b}$ by $2x^a - 5y^b$.

29. $.12x^{3a} - .09x^{2a}y^b - .2x^ay^{2b} + .15y^{3b}$ by $.4x^a - .3y^b$.

30. $2x^{\frac{5}{6}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} - 3x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y^{\frac{5}{6}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{3}}$.

31. $x^{\frac{3}{2}} + xy^{\frac{1}{2}} + x^{\frac{1}{2}}y + y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

32. $-\frac{3}{4}x^4 - \frac{5}{8}x^3 + \frac{119}{60}x^2 - \frac{3}{20}x - \frac{1}{5}$ by $\frac{3}{4}x^2 + x - \frac{4}{5}$.

33. $x^6 + \frac{5}{6}x^4 - \frac{13}{36}x^3 - \frac{1}{6}x^5 + \frac{37}{36}x^2 - \frac{17}{36}x + \frac{7}{24}$ by $x^2 - \frac{2}{3}x + \frac{1}{2}$.

34. $(2x^4 + 3x^2 - 5x + 2 - 6x^3) \div (x^2 - 3x + 1)$.

35. $(6x^3 + 4xy^2 - 13x^2y + 2y^3) \div (2x - 3y)$.

36. $(2x^4 + 6x^2 - 4x - 5x^3 + 12) \div (x^2 - x + 1)$.

37. $(4x^4 - 4x^2 + 1) \div (2x - 2x^2 + 1)$.

38. $(5x^3 - 8x^2 + 8x + 7) \div (5x - 3)$.

39. $(x^4 - x^3 - 7x^2 + 28x - 11) \div (x^2 - 4x + 7)$.

40. $(x^4 - 2x^3 - 10x) \div (x^2 - 3x + 1)$.

41. $x^6 \div (x^2 - 2x + 1)$ to four terms.

42. $(x^7 - 4) \div (x^2 - 1)$ to four terms.

Exercises — Word Problems

1. Write the following by means of algebraic symbols:

(a) Oranges are selling at 3 cents each. What is the cost of o oranges?

(b) Oranges are sold at 2 cents each and at a loss of c cents each. What was the cost of o oranges?

(c) Apples are sold at 5 cents each and c cents profit is made on each apple. What was the cost of one apple? Of a apples?

2. By how much is $4k$ more than $3h$? $3x$ more than $4y$? By how much does x exceed 5? 5 exceed x?

3. A line is 36 inches long. If a piece i inches is cut off, how much remains? If $(i + 2)$ inches were cut off? $(i - 4)$ inches?

4. Two birds fly away from a point in opposite directions, one at the rate of 35 miles per hour and the other at the rate

of 40 miles an hour. How far apart will they be in h hours? In $(h + 2)$ hours? In $(h - 6)$ hours?

5. A horse and a cow together cost \$275. If the horse cost d dollars, how much did the cow cost?

6. George threw a ball 40 feet and his brother threw one f feet farther. How far did his brother throw the ball?

7. If I buy a hat for d dollars and give a \$10 bill in payment, how much change do I receive?

8. A half quire of paper contains 12 sheets. How many sheets are there in h half quires? In x quires? In $2x$ quires? In $.5x$ half quires?

9. There are 5280 feet in a mile. How many feet in $5x$ miles? In $\frac{1}{2}x$ miles? In $.25x$ miles?

10. A book cost c cents to make, o cents for overhead, and then is sold at a profit of p cents. What is the selling price of one? Of 7? Of $9x$?

11. A man sent an article by freight. It cost him 75 cents for the freight and i cents to insure it against damage or fire, and c cents for cartage. The buyer pays x cents for cartage and s cents for its storage for 30 days. He then sells the article at a gain of g cents. What is the selling price of one? Of five?

12. The difference of two numbers is 15. The smaller number is x, what is the larger?

13. The sum of two numbers is 15. One of them is x, what is the other?

14. The product of two numbers is 15. One of them is x, what is the other?

15. The quotient of two numbers is 15. One of them is x, what is the other if there is a zero remainder?

16. A cube is i inches long. What is the total length of all of its edges?

17. A rectangle is l inches long and w inches wide. What is its area? Its perimeter? What would $2\,l(w + 3)$ represent?

18. A rectangle has an area of xy square inches. If the width is y inches, what is its length? Its perimeter?

19. A square is i inches on a side, what is its area? Its perimeter?

20. A square has an area of s^2 square inches. What is the length of a side?

21. What is the multiplier when the product is $9\,x^2y$ and the multiplicand is $3\,y$?

22. What number multiplied by $5\,x$ gives $6\,y$ as a product?

23. How much is left from a \$10 bill after paying for 6 yards of kersey at k cents a yard and 8 yards of cashmere at c cents a yard?

24. Two men are 50 miles apart, and walk toward each other, one at the rate of 2 miles an hour, the other, 3 miles an hour. How far apart will they be at the end of h hours?

25. John bought 10 quarts of milk, and on his way he spilled 3 times q quarts, less 4 quarts. How many quarts had he left?

Exercises

1. The population of a certain town at the present time is P. If it gains $b\%$ of this population by births and loses $d\%$ of this population by deaths regularly each year for n years, what will represent the population at the end of that time?

2. The sides of a rectangular solid are x, $x + 2$, and $x + 3$. What expresses the volume? Find the volume when $x = 3$.

3. The area of a circle may be denoted by the formula $A = \frac{1}{2} Cr$. · If $C = 2\pi r$, find the area of a circle whose radius is 10.

4. If the space s passed over by a body moving with a uniform velocity v in the time t is represented by the formula $s = vt$, what is the space traveled by:

(a) An automobile, when $v = 30$ miles per hour, in 12 hours.

(b) A bicyclist, if $v = 15$ miles per hour, and $t = 3$ hours.

(c) A train in 6 hours, when $v = 45$ miles per hour.

(d) A bird in 3 hours, when $v = 3000$ feet a minute.

5. The volume of a cylinder is denoted by the formula

$$V = \pi r^2 h.$$

If 1 cu. ft. will hold $7\frac{1}{2}$ gallons, how many gallons will a cylindrical tank 10 feet high with a radius of the base of 5 feet hold?

6. A man has a large manufacturing plant which is worth \$15,000. He is taxed on three fourths of its value at the rate of d dollars per hundred dollars. If T represents the tax, write a formula for the tax on his property.

7. The formula used to change Fahrenheit thermometer reading to Centigrade reading is $C = \frac{5}{9}(F - 32)$.

What is the Centigrade reading when the Fahrenheit reading is (a) 113°? (b) 55°? (c) 32°? (d) $-23°$?

8. The value of American exports for three different years was as follows:

One year \$4,409,000.

Five years later \$21,636,000.

One year later \$28,300,000.

Construct a graph by the use of bars to represent the above facts.

9. The value of the imports into the United States for the same years as stated in example 8 was

\$4,910,000, \$2,446,000, \$2,000,000.

Construct a graph to represent these facts.

10. The average income of a group of college graduates for ten years after graduation was

First year	\$ 700	Sixth year . . .	\$2400
Second year . . .	850	Seventh year . .	2300
Third year	1200	Eighth year . . .	2700
Fourth year . . .	1700	Ninth year . . .	3250
Fifth year	2000	Tenth year . . .	3800

Construct a curve graph to represent these facts.

11. The population of Buffalo as given in round numbers at three consecutive ten-year periods was

1900	352,000
1910	423,000
1920	506,000

Represent the above facts by a curve graph.

If we consider the average increase for ten years, continue the curve and estimate the population in 1930.

In 1950. In 1970.

12. At the same place on two successive New Year's days the temperature readings were as follows:

	Midnight	4 A.M.	8 A.M.	Noon	4 P.M.	8 P.M.
1st New Year's	8°	5°	7°	12°	14°	12°
2d New Year's	− 4°	− 6°	− 4°	0°	2°	− 1°

Draw two curves on the same chart to show the above facts.

At what hour was there the greatest difference of temperature?

13. The gross revenue of a certain telephone system was disposed of as follows:

Salaries and wages . . .	50%
Interest	19%
Surplus	6%
Taxes	5%
Material, rent, traveling .	20%

Represent the above data on a chart. Use a rectangle and let its length represent 100%.

14. The earnings and expenses of a certain railroad by years are given in the following table:

	1915	1916	1917	1918	1919	1920
Earnings	$400,000	$600,000	$680,000	$800,000	$960,000	$1,000,000
Expenses	260,000	610,000	460,000	500,000	640,000	680 000

Construct the two curves on the same diagram.

When do the two curves cross?

What does this mean?

Which curve should be at the top for practically all of the time? Why?

CHAPTER II

THE EQUATION

You have learned in your earlier work in algebra how valuable the equation is in all applied work. The ability to express relations between magnitudes in equation form and to solve the equation is the essence of algebra. You can not be too careful, therefore, in endeavoring to obtain correct ideas of this important topic.

18. An Equation is a statement that two algebraic expressions are equal. These expressions are called the *members of the equation*. Equations are of two types, namely, *Identical Equations* and *Conditional Equations*.

The identical equation or identity is true for all values of the letters concerned. Examples: $2 \cdot 3 = 6$,

$$(a + b)(a - b) = a^2 - b^2, 2x + 3x = \frac{10}{2}x.$$

When we desire to emphasize the identical character of an equation, we use the sign $\equiv$. This sign is read "is identical with."

The conditional equation is true only for a limited number of values of the letter concerned. Examples: $2x - 3 = 5$, which is true only when $x = 4$; and $x^2 - x = 6$, which is true only when $x = 3$ or -2.

NOTE. *All numerical equations are identities.*

19. Roots of Equations. — A root of an equation is that value of the unknown quantity which makes the equation

true. Only conditional equations have roots. In the illustrations of the preceding paragraph the root of the first equation is 4, and the roots of the second equation are 3 and -2.

You should note that when the root is substituted for the unknown quantity in a conditional equation, it is reduced to a numerical identity.

20. Simple Equation. — A simple or linear equation in one unknown quantity is an equation which contains that unknown quantity to the first power only.

21. Solution of Equations. — The solution of an equation is the process of obtaining the root or roots. In simple linear equations you will accomplish this by the aid of certain *axioms*, which are already familiar to you both in algebra and geometry. They are the axioms of addition, subtraction, multiplication, and division, respectively, and may be stated as follows:

1. If the same number be added to both members of an equation, the equation remains true.
2. If the same number be subtracted from both members of an equation, the equation remains true.
3. If both members of an equation be multiplied by the same number, the equation remains true.
4. If both members of an equation be divided by the same number (except zero), the equation remains true.

Illustrative example I: Solve $x - 5 = 10$.

Solution: Adding 5 to each member of the equation according to the first axiom, we obtain $x = 15$.

Illustrative example II: Solve $x + 6 = 8$.

Solution: Subtract 6 from each member of the equation according to the second axiom and we obtain $x = 2$.

Illustrative example III: Solve $\frac{1}{2}x = 8$.

Solution: Multiply each member of the equation by 2 according to the axiom of multiplication and we obtain $x = 16$.

Illustrative example IV: Solve $7x = 21$.

Solution: Divide each member of the equation by 7 according to the axiom of division and we obtain $x = 3$.

Illustrative example V: Solve $4x - 3 = 2x + 5$.

Solution:

$4x = 2x + 8$. Axiom of addition.
$2x = 8$. Axiom of subtraction.
$x = 4$. Axiom of division.

Oral Exercises

Solve each of the following equations and state which axiom or axioms are used in each case:

1. $3x = 9$.
2. $x - 5 = 8$.
3. $x + 6 = 2$.
4. $\frac{x}{3} = 6$.
5. $2x + 1 = 5$.
6. $9x - 3 = 15$.
7. $4x = x + 6$.
8. $2x = 9 - x$.
9. $\frac{1}{4}x = 3$.
10. $2x = \frac{x}{2} + 6$.
11. $4x - 2 = 3x + 5$.
12. $5x = x - 10 - 2x$.
13. $1 + 3x = x - 11$.
14. $.5x = 25$.
15. $6x = .54$.
16. $8 = 4x$.

17. Give an illustration to show that the axiom of division fails when the divisor equals zero (0).

18. Point out the error in the following argument:

Let $a = b$.
Then $a^2 = ab$. Axiom of multiplication.
Then $a^2 - b^2 = ab - b^2$. Axiom of subtraction.
$(a + b)(a - b) = b(a - b)$. Factoring.

Then	$a + b = b.$	Axiom of division.
But $a = b.$	$\therefore a + b = 2\,b.$	
Hence	$2\,b = b.$	
	$\therefore 2 = 1.$	Axiom of division.

22. Transposition is the process of omitting a term from one member of an equation and writing the same term with the *opposite sign* in the other member. *It is equivalent to the addition and subtraction axioms.*

Illustrative example I: Solve $x - 5 = 8$.

Solution: By transposing the -5 we obtain

$$x = 8 + 5 \text{ or}$$
$$x = 13.$$

This is equivalent to adding 5 to each member of the equation.

Illustrative example II: Solve $2\,x + 3 = 11$.

Solution: By transposing $+3$ we obtain $2\,x = 11 - 3$ or $2\,x = 8$ or $x = 4$.

This is equivalent to subtracting 3 from each member of the equation.

We should think of transposition, not as a process of algebra, but merely as a short way of applying the axioms of addition and subtraction. It is especially useful when several terms are to be transposed.

Rule. — (1) *To solve a simple equation containing one unknown quantity, transpose to the left member of the equation each term containing the unknown quantity, and to the right member each term not containing it.*

(2) *Collect similar terms in each member.*

(3) *Divide each member by the coefficient of the unknown quantity.*

23. Check of Solutions. — You should remember that the final test as to whether a quantity is or is not a root of an equation is whether it does or does not cause the original equation to reduce to a numerical identity when it is substituted for the unknown quantity. If it does so, it is said to *satisfy* the equation and is a root. To be a valid check the substitution must be made in the *original* equation.

24. Literal equations are equations in which some or all of the known quantities are represented by letters. It is customary to use the last letters of the alphabet to represent unknown quantities and the earlier letters to represent known quantities. This custom has been followed in this book except as otherwise stated. Formulas in which the letters used are initial letters are exceptions to this rule.

Illustrative example I:

Solve and check: $3x - 10 + x = 5 - 2x + 3$.

Solution:

Transposing, $3x + x + 2x = 5 + 3 + 10$.

Collecting terms, $6x = 18$.

$\therefore x = 3$.

Check: $9 - 10 + 3 = 5 - 6 + 3$.

$2 = 2$ (an identity).

Illustrative example II:

Solve and check: $2K - x = 5x - 10K + 6$.

Solution:

Transposing, $-x - 5x = -10K + 6 - 2K$.

Collecting terms, $-6x = -12K + 6$.

$\therefore x = 2K - 1$.

Check: $2K - (2K - 1) = 5(2K - 1) - 10K + 6$.

$2K - 2K + 1 = 10K - 5 - 10K + 6$.

$1 = 1$ (an identity).

Oral Exercises

1. What is a conditional equation? An identity? Give an example of each. Which kind of equation can be solved?

2. What condition makes $2x = 10$? $x - 1 = 6$?

Is there any value of x such that $(x + 1)^2$ will not equal $x^2 + 2x + 1$? Then $(x + 1)^2 = x^2 + 2x + 1$ is what kind of an equation?

3. What is a simple or linear equation?

4. What is a root of an equation?

5. In the equation $2x - 1 = 9$, is 3 a root? How can you tell?

6. What do you mean when you say that a certain number "satisfies an equation."

7. What is transposition?

8. State the rule concerning signs in transposition.

9. When a positive term is transposed, what axiom have you used? A negative term?

10. Why is it important to check the solution of equations?

11. Why do we always check in the original equation?

12. If you are asked to solve $a - 2y + 3m = 6m - y + 5a$, what letter do you solve for? Why?

13. What exception is there in the use of the division axiom?

14. State an equation whose solution would require the use of the multiplication axiom.

15. What is the origin of the word "algebra"? See "First Course in Algebra," page 70.

16. What value satisfies $9 = \frac{1}{3}x$? $17 = \frac{1}{2}x$? $20 = .25x$? $60 = \frac{2}{3}x$?

Exercises

Solve and check each of the following equations:

1. $6x - 10 = 2x + 14$.
2. $8 - 5x = 3x + 8$.
3. $2x - 1 + 6x = 5x - 17 + x$.
4. $3y + 2y - 5 = 15 - 2y + 4 - y$.
5. $10z - 5 = 2z + 12 + 3$.
6. $2x - 3K = 5x + 12K - 3$.
7. $6x + m - x + n = 6m + 6n$.
8. $mx - 6m^2 = 4mx + 3m^3$.
9. $6x - 2(x + 3) = 5(2x - 1)$.

Hint: First remove parentheses. $6x - 2x - 6 = 10x - 5$, etc.

10. $x^2 - (x - 1)(x + 2) = 4x + 7$.
11. $(x - 3)(x + 6) - (x + 1)(x + 4) = 6x - 10$.
12. $2x(x - 3) = (2x + 5)(x - 3)$.
13. $(3x - 1)(1 - x) + 3x^2 = 4x - (2x + 6)$.
14. $(x - 3)(x + 2) + (2x - 1)(x - 5) = (3x - 6)(x + 2)$.
15. $x^2 - (x + 1)(2x + 1) = x(2 - x)$.
16. $5(x - 3) + 3(8 - x) + 29 = 7(6 - x) + 50$.
17. $y^2 + 8y - 5 = y(y - 1) + 5(y + 3)$.
18. $3(x-7)+5(x+4)(x+2) = x^2+4(x+3)(x+1)+10$.
19. $3[x + x(x - 3) + 4] = (3x - 2)(x + 1)$.
20. $(x - 2)(x - 4) + (x - 5)(x - 3) = 2(x - 3)(x - 5)$.
21. $(x - 2)^2 + 3(x - 4)^2 = 4(x + 3)^2$.
22. $(2x + 3)(3x - 2) = x^2 + x(5x + 3)$.
23. Does $x = 1, 2$, or 3 satisfy the equation
$(3x - 2)(x + 4) - 7 = (x - 1)(2x - 3) + x^2 + 12$?

24. What value of x satisfies

$$3(6 - x) + 4(3x + 5) + 3(6 - x) = 4(3 + x)?$$

25. Solve: $h + 30 + 7x = 8 - 6h + 6x - d + b$ for x.

26. What roots are lost if we divide each member of the following equations by the expression set at the left?

(a) $x - 4$. $\quad x^2 - 16 = 0$.

(b) x. $\quad x^3 + 5x^2 + 6x = 0$.

(c) $x - 5$. $\quad x^2 - 25 = 0$.

(d) $x + 2$. $\quad x^2 + 4x + 4 = x + 2$.

27. What extra (called extraneous) root is introduced into the equation $x = 6$ if both members are multiplied by x? By $x - 3$? By $x + 2$?

28. What root is lost if both members of the equation $x^2 - 9 = x - 3$ are divided by $x - 3$?

29. Show which of the following quantities, 2, 3, -4, is a root of the equation $(x + 4)(x - 9) = x(x + 4)$.

30. Solve $105.2 - 64x = 26x - 164.8$ for the value of x.

25. Algebraic Expression.—As a preliminary to the solution of problems it is necessary for you to be able to think in terms of algebraic symbols. You should have acquired this ability in your first course in algebra. The following exercise will give you practice in this form of expression.

Oral Exercises

1. Represent 5 consecutive whole numbers if the smallest is n; if the largest is n; if n is the middle one.

2. Represent 3 consecutive odd whole numbers if x is the smallest.

3. Why should $2n$ represent an even number rather than an odd number?

4. If John is x years old now, how old was he 5 years ago? n years ago? How old will he be 10 years hence? m years hence?

State as an equation the fact that John's age 10 years hence will be twice his age 5 years ago.

5. If I travel 30 miles an hour, how far will I go in 5 hours? In h hours? If my rate is K miles an hour, how far will I go in h hours?

6. Rate multiplied by time gives what?

State several illustrations.

$$\frac{\text{Distance}}{\text{Rate}} = ? \qquad \frac{\text{Distance}}{\text{Time}} = ?$$

7. Express 3 yards, 2 feet, 5 inches as inches.

Express the sum of y yards, f feet, and i inches as inches; as yards.

8. Express the sum of x dollars and y cents as dimes.

9. What is meant by 5%? What is 5% of \$80?

10. What is y% of \$80? y% of \$$x$?

11. If a man travels y miles in x hours, how far will he go in 1 hour? In n hours?

12. If a quantity of grain will last n cows a week, how long will it last 1 cow? m cows?

13. A man bought goods amounting to d dollar x dimes, and y cents. He presented a check for k dollars. How many cents change did he receive?

14. If I exchange k dozen eggs for y yards of cloth worth x cents per yard, what is the price of the eggs per dozen?

15. Does 23 mean $2 \cdot 3$? Does it mean $2 + 3$? Does it mean 2 tens + 3 units or $10 \cdot 2 + 3$?

16. How would you write the number whose left-hand digit is x and whose right-hand digit is y?

17. Write the number expressed by the same digits reversed.

18. x yards per second is how many miles per hour?

19. If the greater of two numbers is 1 more than twice the smaller, and the smaller number is x, what is the greater? If the greater number is x, what is the smaller?

20. In an orchard, there are a apple trees and p pear trees. How many trees are there in all?

21. An organ grinder received a cents at one house, b cents at another house, and c cents at a third house. What was the value of his money in dimes?

22. There are 5280 feet in a mile. How many feet does a boy walk if he walks m miles and 575 feet? How many miles?

23. How many dollars will a man save in w weeks if he saves 50 cents a week?

24. If your pulse beats 4320 times in one hour, how many times will it beat in d days of 24 hours each?

25. A clerk received \$20 a week. If he spent \$$x$ for board and \$$y$ for other expenses, how many dollars did he save in w weeks?

26. Problems. — The ability to solve verbal problems is a gauge of your mastery of algebra and of your ability to apply it. It is therefore very valuable, especially for the more advanced pupil.

Since each problem is different from the others, no definite rules for the solution of problems can be laid down. The following suggestions are offered:

1. Read the whole problem very carefully and try to understand its meaning thoroughly before you attempt to solve it.

2. Represent the quantity you are to find by x or some other convenient letter.

3. If you are to find more than one quantity, represent the smallest one by x, and represent the related quantities in terms of x.

4. Express in symbols as an equation that which the problem states in words.

5. In checking your solution, always check in the problem rather than in the equation.

Exercises

1. Find 3 consecutive numbers whose sum is 17 more than the largest number.

2. Find 3 consecutive odd numbers whose sum is 5 more than twice the next consecutive odd number.

3. The greater of two numbers is 5 less than twice the smaller, and their sum is 28. What are they?

4. In thirty years James will be 8 times as old as he was 12 years ago. How old is he now?

5. Henry's father is now twice as old as he is; 16 years ago he was four times as old. How old is each now?

6. Two men start from points 176 miles apart and travel toward each other until they meet. Find the rate of each if one travels two miles an hour faster than the other and they meet in 8 hours.

7. A marksman hears a bullet strike the target 1650 feet away in $4\frac{1}{2}$ seconds. Find the rate of the bullet in feet per second if the velocity of sound in air is taken as 1100 feet per second.

8. 6 hours after a boy starts from a certain point on a bicycle at the rate of 12 miles an hour he is followed by a man with an automobile and is overtaken in 8 hours. What is the rate of the automobile?

9. How far may a man walk into the country at the rate of 4 miles an hour, if he rides back at the rate of 20 miles an hour and must be back in 8 hours?

Hint: Let x = number of hours walking out.

10. Mary is 24 years old. Mary is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

11. A man was hired for 30 days at \$3 per day and board for every day that he worked. Each day that he was idle he was to pay \$1 for his board. At settlement he received \$66. How many days was he idle?

12. Mr. Jones had 60 cattle and Mr. Brown had 45. Mr. Jones bought a certain number from Mr. Brown, after which he had 6 times as many cattle as Mr. Brown. How many did he buy?

13. A man bought 60 bushels of potatoes for \$104. For one kind he paid \$1.80 a bushel, and for the other kind he paid \$1.60 a bushel. How many bushels of each kind did he buy?

14. I have twice as many dimes as nickels and 10 more quarters than the number of dimes and nickels together. If I have \$8.50 in all, how many coins of each kind have I?

15. Eight boys agreed to buy a tent, but two of them were unable to pay, so that each of the others had to pay \$4 more than he had expected to pay. What was the cost of the tent?

16. What number is that, from which 6 being subtracted, and the remainder multiplied by 11, the product will be 121?

17. After A had loaned 10 dollars to B, he lacked 8 dollars of having as much money as B; and together they had 60 dollars. How much had each at first?

18. A person has 6 hours at his disposal. How far may he ride into the country at 8 miles an hour, that he may return home in time, walking back at the rate of 4 miles per hour?

19. What number must be added to each of the numbers 2, 3, 6, 5 in order that the sum of the squares of the first two numbers may be 83 less than twice the product of the last two?

20. A room is 5 feet longer than its width, and if we decrease the length by 3 feet and increase the width by 4 feet, the area will be increased by 16 square feet. What are the dimensions of the room?

21. If we increase each side of a square by 3 feet, we shall increase the area by 33 square feet. Find a side of the original square.

22. If we decrease each side of a square by 5 feet, we shall decrease the area by 45 square feet. Find the side of original square.

23. The sum of twice one acute angle of a right triangle and three times the other acute angle is 192°. How many degrees in each of the acute angles?

24. The difference of two consecutive angles of a parallelogram is 20°. How many degrees in each angle?

25. If we increase two opposite sides of a square by 4 feet and the other two opposite sides by 6 feet, we shall then have a rectangle whose area is 74 square feet more than the square. Find a side of the square.

26. The first angle of a triangle is 4 times the result of decreasing the second angle by 15°. The third angle is 3 times the second angle. How many degrees in each angle?

27. A man's property is insured in three different companies. The first company insures it for $10,000. The second company insures it for $4000 more than twice the

amount of insurance in the third company. If the property is insured for $29,000, how much is carried in the second and third companies?

28. A man has two investments totaling $10,000. On one he receives 8% interest and on the other he receives 4% interest. The 8% investment yields an annual income of $320 more than the 4% investment. How much has he invested at each rate?

29. At a certain baseball park the reserved seats cost $1 and the other seats were 50 cents each. If 3600 tickets were sold in all and the total amount received was $2225, how many tickets of each kind were sold?

30. Fifty-five coins, consisting of dollars and half dollars, amount to $36. How many coins of each kind are there?

31. A and B have $50 between them. A wins from B as much as A had originally, and he then has three times as much as B. How much had A at first?

32. If one number exceeds another number by 3 and the square of the larger exceeds the square of the smaller by 99, what are the numbers?

33. It is claimed that 200 times as many lives were lost in an earthquake in Japan in 1923 as were lost in an earthquake in San Francisco in 1906. If 100,500 lives were lost in both, how many were lost in each?

34. If the product of two consecutive even numbers is increased by 28, it will equal the square of the larger number. Find the two numbers.

35. An automobile leaves New York for Chicago on a road paralleling a railroad track at a rate averaging 26 miles per hour. Six hours later an express train leaves for Chicago, averaging 50 miles per hour. In how many hours will the train overtake the automobile?

36. A new U. S. dirigible is said to be able to make 60 miles an hour and an airplane is making 80 miles per hour. If the airplane is 60 miles behind the dirigible, in how many hours will it overtake the dirigible?

37. In the olden days a pony express, traveling at the rate of 60 miles a day, had been gone 5 days, when a second pony express was sent after them, traveling at the rate of 75 miles per day. In what time would the one overtake the other?

38. A man starts for a distant point at 6 A.M. at the rate of 4 miles per hour. Six hours later another man starts from the same place in the same direction at the rate of 20 miles an hour. When will the second man overtake the first man?

39. A man paid his bill of \$16.75 with quarters and half dollars. If he had 60 coins in all, how many had he of each kind?

40. A man invests \$1200, part at 5% and part at 4% per annum, and his total income is \$55.50. How much did he invest at each rate?

Exercises — Review

1. Add $6x^2 - 5xy + 4z$, $12x^2 + 6xy - 4z$, $13xy$, $3x^2 - 14z$, $7xy - 17z$.

2. Subtract the product of $-7x^2$ and $4yz$ from the product of $-6yz$ and $-3x^2$.

3. Multiply $4x^2 + 6.3x - 7$ by $.2x - 3$, and check.

4. Divide $x^{3a+2} - 2x^a y^{a+1} - x^{2a+2}y^{2a} + 2y^{3a+1}$ by $x^a - y^{2a}$, and check.

5. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y) = ?$

6. Subtract $bx^2 + h$ from 0 and add that result to $ax^2 + h$.

7. If I receive d dollars my first year and a regular annual increase of x dollars thereafter, what is my salary after y years? Write your answer as a formula, letting S represent the last salary.

8. Given $d = \$1200$, $x = \$100$, $y = 12$ years. Find S in the above formula.

9. Express each of the following as equations:

(a) The sum of 15 and h equals $3x$.

(b) The double of x plus one fourth of y equals 75.

(c) Five subtracted from $3x$ gives the same result as 9 subtracted from x.

(d) Four times the square of a number one larger than x exceeds the number by 7.

10. Solve $1.2x - .2(.18x - .05) = .4x + 8.9$.

11. A father's age is three times that of his son, and in 10 years it will be twice as great. How old is each?

12. Prove that

$$(a+2)(b+2) - 4(a+1)(b+1) + 6ab - 4(a-1)(b-1) + (a-2)(b-2) = 0.$$

13. Show that if $x = y + .3$, then $x^2 + x$ will equal $y^2 + 1.6y + .39$.

14. What is the interest on \$5000 for y years at $r\%$ simple interest?

15. Simplify $4x^2 - [.5x^2 - \{.7x^2 - (.8x^2 - .005)\}]$.

16. Solve and check:

$$.3x - 2(.06x - 5) = 28.$$

17. The difference of the squares of two consecutive even numbers is 48. Find the numbers.

18. A man has \$5.95 in quarters and dimes. He has 31 coins in all. How many has he of each kind?

19. If s denotes the size of a shoe and f denotes the length of the man's foot, form a table and draw a graph to show the relation of the sizes from 6 to 12 if $f = \frac{s}{3} + 8$.

20. If the tickets to an entertainment cost 50 cents for adults and 25 cents for children, and 200 tickets are sold amounting to \$80, how many of each kind are there?

21. Which of the following are identities?

(a) $\dfrac{x^2 - y^2}{x + y} = x - y.$

(b) $(x + y)^2 = x^2 + 2xy + y^2.$

(c) $x^2 - 3x + 2 = (x - 1)(x - 2).$

(d) $x^2 - 4x = (x - 8)(x + 8).$

(e) $x^2 = (x + 4)(x - 7).$

What is an identity? An equation?

22. Make six graphs from the following information. Each chart is to contain the same kind of information about each salesgirl.

	Mary	Lucy	Anna	Ruth	Agnes
Number of customers	48	36	54	102	86
Number of buying customers	44	30	27	84	85
Number of sales	85	69	86	101	94
Number of returned sales	14	0	18	12	0
Value of sales	\$258	\$315	\$225	\$198	\$98
Value returned sales	\$70	0	\$69	\$35	0

What might you learn from these facts?

23. Business men often go to great expense to find out which is the best location for a new store. One method employed is to make a count of the number of persons passing certain corners for a certain interval of time (say 15 minutes) at various times of the day. From the following facts, con-

struct curve charts to represent these numbers, and then determine what you can from them.

Hours	Corner #1 Count	Corner #2 Count	Corner #3 Count	Corner #4 Count
8–10	111	120	210	125
10–12	193	202	284	269
12– 2	360	390	330	250
2– 4	450	354	375	148
4– 6	385	480	535	316

24. Express the equation and solve it if $37\frac{1}{2}\%$ of a number equals 81.

25. If $x + .25\,x = 1.25\,x$, then $1.25\,x$ is said to represent x increased by 25% of itself. What do each of the following represent: $1.08\,x$, $1.75\,x$, $1.66\frac{2}{3}\,x$, $1.5\,x$, $x - .25\,x$, $.75\,x$, $.8\,x$?

26. If $8 \neq 10$, make it a true statement. Do the same for $6\,x \neq 9\,x$. For $9\,y \neq 15\,y$.

27. If t represents tens and u represents units, what does $5\,t + 8\,u$ represent? $8\,t + 3\,u$? $12\,t + u$?

CHAPTER III

PRODUCTS AND FACTORS WITH APPLICATIONS

PROFICIENCY in algebra requires not only that you should be able to multiply and divide polynomials, but that you should be able to do this mentally in simple cases. It is also necessary to recognize a product when you see it, and to be able to break it up into the factors of which it is composed. Without this ability much of the later work with fractions and equations would not be possible.

27. The Factors of a product are those quantities which when multiplied together produce that product. A quantity is said to be *prime* when it has no factors except itself and 1. *Prime factors* are factors which are themselves prime.

Thus the factors of 24 are 4 and 6 or 2 and 12 or 3 and 8, but these sets of factors are not prime, since 6, 12, and 8 can all be factored further. The prime factors of 24 are 2, 2, 2, and 3. In all work in factoring prime factors are required.

28. Product of a Polynomial by a Monomial. — This case has been discussed in section 10, but is included here for convenience.

Exercises

Perform each of the indicated multiplications:

1. $-2\,a^2(a^2 - 3\,ab - 6\,b^2)$.
2. $5\,ab(a^3 - a^2b + 3\,b^2)$.
3. $m^2n^2p(2\,m^2 - 6\,mnp + 5\,np^2)$.
4. $6\,x^2z(3\,x^2 - 2\,xyz - 4\,yz^2)$.

5. $-21\,m(m^2 - lmnp - 6\,np^2 + lp^3)$.

6. $8\,ap^2(2\,a^2p - 5\,ar + 6\,pr^2)$.

7. $x(x + y + \frac{1}{4}\,y^2)$.

8. $.8(x^3 - 16\,x + 24\,x - 5)$.

9. $2\,HK(H^2 + 5\,HK - 6\,K^2)$.

10. $2\,ax^2(10\,ax - 6\,a + 4\,a^2 - a^3)$.

11. $-16\,ab(ab - \dfrac{3\,c}{4} + \dfrac{3\,x^2y^2}{b} - 4\,abc^2)$.

12. $.6\,x^n(1 - 1.2\,x^n + 1.8\,x^{2n})$.

13. $4.8\,x^ny(x^{2n} - 9.6\,x^{3n}y^4 + 14.4\,x^{4n}y^5)$.

14. $\frac{1}{2}\,xy(2\,xy^2 - 6\,x^2y - 12\,x^3)$.

15. Simplify $6 + 8(.4 - .05)$.

16. Simplify $7 + 9(.8 - 0 + .02)$.

17. $.3(x^2 - .12\,x) = ?$

18. $\frac{2}{5}\,\pi(R^2 - r^2) = ?$

19. $x^{n+1}(x - 3\,x^n + 6\,x^{n-1}) = ?$

20. What is the value of $x^n(x^n - 3\,x^{2n} - 5\,x^{3n})$ when $x = 3$ and $n = 2$?

29. Expressions Having a Monomial Factor. — The expressions of this exercise are such as arise from a multiplication of the type found in the preceding exercise. Since the monomial multiplier multiplies each term of the polynomial, it can be recognized in each term of the product and removed as a factor.

Example. Factor $6\,a^3x^4 - 15\,a^2x^2y + 21\,a^5x^5y^2$.

Solution: Inspection of the terms of this polynomial shows that $3\,a^2x^2$ is contained in each. Hence it is a factor. We divide the polynomial by this monomial factor, term by term, to obtain the polynomial factor.

We obtain $3\,a^2x^2(2\,ax^2 - 5\,y + 7\,a^3x^3y^2)$.

Exercises

Factor each of the following:

1. $2\,p^2 - 6\,pt - 8\,p^3q - 16\,pq^3$.
2. $a^4x^3p - 3\,a^3x^2p^2 - 4\,a^2x^3p - 5\,a^3x^2p^6$.
3. $21\,x^2p^2y^2 - 49\,x^3p^2y^3 - 63\,x^2p^3y^4$.
4. $4\,mn - 8\,m^2n^2 - 16\,mn^3 + 12\,m^4n^2$.
5. $-3\,a^2b^3c^5d^8 - 15\,a^3b^4c^7d^4 + 21\,a^4b^2c^7d^5e$.
6. $33\,x^2p^2z^2 - 66\,x^3p^3z + 77\,x^5p^6z^2 - 11\,x^6p^8$.
7. $16\,x^2(m+n)p^2 - 12\,xy(m+n)^2p^3 - 20\,x^3y^3(m+n)^3p$.
8. $17\,x^3(x-y)^3 + 51\,x^5(x-y)^2 - 85\,x^2(x-y)^4$.
9. $4\,x(a+b+c) + 8\,x^2(a+b+c) - 12\,x^5(a+b+c)$.
10. $5\,x^3 - (6\,x^2 + 4\,x^4 - 9\,x^5)$.
11. $1.5\,x^6 + 3\,x^4 - 2\,x^3 + 3.5\,x^2 - 4.5\,x + .5\,x)$.
12. $3\,x^{2n+2} + 12\,x^{3n+3} - 18\,x^{4n+4} + 24\,x^{5n+5}$.
13. $\frac{1}{4}\,x^3 - \frac{1}{8}\,x^2 + \frac{3}{4}\,x^5$.
14. $\frac{1}{3}\,x(a+b) + \frac{1}{3}\,y(a+b)$.
15. Compute the value of $189\,\pi - 56\,\pi$ by factoring when $\pi = \frac{2}{7}$.
16. Collect the coefficients of x and y in
$$2\,ax + 2\,ay + 3\,x + 3\,y.$$
17. Collect the coefficients of y^3 and y in
$$3\,y^3 + 4\,y - 5\,y^3 + 7\,xy.$$
18. Compute 56 times \$4.43 + 44 times \$4.43 − 50 times \$4.43.

30. Binomial Factors by Grouping. — Let it be required to factor $l(x+y) - n(x+y)$.

Solution: By means of parentheses this has been grouped into two terms. $(x+y)$ is now contained in each term. If

12. Mr. Carey can do a piece of work in 10 days. After working 3 days on it he is joined by Mr. Smith and they finish the work in $2\frac{1}{3}$ days. In how many days can Mr. Smith do the work alone?

13. A boy buys marbles at the rate of 5 for 2 cents. He sells half of them at 2 for a cent and the other half at $\frac{1}{3}$ of a cent each and gains 10 cents. How many marbles had he?

14. A sum of \$1500 was to be divided equally among a certain number of persons. If there had been 5 persons more, each would have received only two thirds as much. How many persons were there?

15. A man can walk 4 miles an hour. How far may he walk into the country and ride back on a trolley car which runs 20 miles an hour, if he must be home in 3 hours?

16. What number must be added to the numerator and the denominator of the fraction $\frac{1}{2}\frac{2}{3}$ in order that the resulting fraction may equal $\frac{1}{2}$?

17. The numerator of a certain fraction is 7 less than the denominator. If 4 is subtracted from the numerator and 1 added to the denominator, the resulting fraction equals $\frac{1}{3}$. Find the fraction.

18. A man invests a part of \$1800 at 5% and the rest of it at 6%. His total income from both investments is \$100. How much had he invested at each rate?

Hint: Let x = amount invested at 5%.

Then $.05\,x + .06(1800 - x) = 100$.

19. Two thousand dollars of Mr. Smith's income is exempt from taxation. He pays a tax of 2% on the balance. If this tax amounts to \$128, what is his income?

20. A man invests \$7000 in building a home. Two thirds of the remainder of his money is invested in 6% bonds bought

at par ($100), and the balance in a mortgage paying 5%. His total income from these investments is $510. How much money has he?

21. A passenger train traveling 50 miles an hour leaves a certain station $2\frac{1}{2}$ hours after a freight train, and overtakes it in $6\frac{3}{7}$ hours. Find the rate of the freight train in miles per hour.

22. A man rows up a stream $4\frac{1}{2}$ hours and finds that it requires only $2\frac{1}{2}$ hours to make the return trip. If he can row $3\frac{1}{2}$ miles an hour in still water, what is the rate of the current?

23. The circumference of a rear wheel of a certain wagon is 3 feet more than the circumference of the front wheel. The rear wheel makes 100 less revolutions than the front wheel in traveling a distance of 6000 feet. Find the circumference of each wheel.

24. At what time after 4 o'clock are the hands of a clock together?

Hint: The minute hand travels 12 times as fast as the hour hand. Hence while the minute hand travels x minute spaces the hour hand travels $\frac{x}{12}$ minute spaces.

Hence $x = 20 + \frac{x}{12}$.

25. At what time between two and three o'clock are the hands of a clock opposite each other?

26. At what time between 7 and 8 o'clock are the hands of the clock at right angles?

27. At what time between 9 and 10 o'clock are the hands of the clock equally distant from 12?

28. To a mixture of 10 quarts which is $\frac{3}{10}$ alcohol, I wish to add enough water so that the resulting mixture shall

contain 20% alcohol. How many quarts of water must be added?

Hint: Let x = number quarts water to be added.
$x + 10$ = number quarts in the mixture.
3 = number quarts alcohol.

Therefore $\dfrac{3}{x+10} = \dfrac{1}{5}$.

29. If 20 pounds of sea water contains 1.6 lb. of salt, how many pounds of pure water must be added to produce a mixture containing 5% salt?

30. If water is to be added to 40 quarts of milk containing 4% of butter fat so as to produce a mixture containing 3% butter fat, how many quarts of water must be added?

31. A chemist has 10 quarts of a 10% solution of acid. He wishes to add enough water to reduce it to a 4% solution. How many quarts of water must be added?

32. Gold, when weighed in water, loses $\frac{1}{19}$ of its weight, and silver, when weighed in water, loses $\frac{2}{21}$ of its weight. If a mixture of gold and silver weighing 40 pounds in air loses 3 pounds when weighed in water, how many pounds of gold and silver are there in the mixture?

33. A and B have \$100 between them. If A spends $\frac{1}{3}$ of his share, and B spends $\frac{1}{4}$ of his share, they will then have \$70 left. How much had each at the beginning?

34. A man spends one third of his money and \$5 more. He then has left one fifth of his money and \$35. How much had he at first?

Word Problems

1. Given the rate of discount as $d\%$ and the marked price as m dollars, find the discount and the selling price.

2. How many times is $\dfrac{a}{b}$ contained in $\dfrac{c}{d}$?

3. How many halves in two fourths? In f fourths?

4. What is one half of one third? One half of t thirds?

5. What is the difference between one half of one third and one third of one fourth? Between one half of t thirds and one third of f fourths?

6. A had t thirds of a dollar and gave B one fifth of it. What part of a dollar did B receive?

7. What is two thirds of e eighths?

8. Express D in terms of d, q, and r when D equals the dividend, d equals the divisor, q equals the quotient, and r is the remainder.

9. At c cents a pound, how many pounds of meat can I buy with d dollars?

10. What is the average of the two numbers a and b? Of the three numbers a, b, and c? Of n numbers?

11. How many yards of cloth at c cents a yard can I buy with 5 dollars? With d dollars? With x dimes?

12. Find half the perimeter of a rectangle which is f feet long and i inches wide, in terms of feet. In inches.

13. What is the length of a rectangle which is $3\,w$ inches wide, if the perimeter is $24\,w$ inches? If the perimeter is $19\,w$ inches?

14. By how much does x exceed 60? 60 exceed x?

15. What value of x makes $6\,x$ equal to 48?

16. If a man can walk 3 miles an hour, how many hours will he require to walk x miles?

17. Find the price in cents of x oranges at 20 cents a dozen.

18. State the proof of $ax \div a$ equals x.

19. How much can a man earn in 5 months at d dollars a month? In one year?

20. What is the equation which expresses the relation that the sum of a, b, and c divided by two times their product is equal to $\frac{1}{4}\frac{7}{8}$ of their sum diminished by b?

21. What is the proof of the statement that a times x plus b times x equals x times the quantity a plus b?

22. If y yards of cloth cost $5\ \frac{3}{5}$ dollars, what is the price of 1 yard?

23. If p pounds of sugar cost 15 cents, what is the price of 1 pound? Of 5 pounds? Of x pounds?

24. How many times 7 are 21 twenty-thirds? Is x?

25. If 7 pounds of cheese cost \$1.89, what is the price of 1 pound? What is the cost of x pounds at the same rate?

26. How many times 4 are 8 times 7? Are x times y?

27. If 2 apples cost 5 cents, what is the price of 1 apple? What is the cost of a apples at the same rate?

28. If you divide d dollars equally among 5 persons, how much will each have? How many cents?

29. How many cents will $\frac{2}{3}$ of a pound of candy cost if 1 pound cost 6 dimes? How many cents would x pounds cost?

30. If 15 fountain pens cost \$30, what will 5 of them cost? What would x of them cost at the same rate?

31. $4\frac{1}{2}$ times 7 is one half of what number? Is f fifths of what number?

32. By how much does $3x - 6$ exceed $x + 4$?

33. A man has x dollars and d dimes. He spends c cents. How many cents has he left?

34. What must be added to $3x - 5$ to produce $2x - 3$?

35. y years ago a man was 25 years old. How old is he now?

36. How many hours will it take a man to walk n miles at the rate of m miles an hour? How far would he walk in 6.5 hours?

37. If the denominator of a fraction is 4 more than the numerator, what expresses the denominator if n represents the numerator? What represents the fraction?

38. I gained d dollars by selling an article for s dollars. What represents the cost?

39. If one half of a line is f feet, what is the length of the line in yards? In inches?

40. What is the cost of 8 apples if a apples cost 24 cents?

Review Exercises

1. Write the following as an equation and solve it for the value of x.

x divided by 9 gives 12 as a quotient and 1 as a remainder.

2. From the sum of $y(3h + 4k)$ and $y(2h - 3k)$ subtract their difference.

3. Solve and check:

$$\frac{5x + 2}{3} - \left(3 - \frac{3x - 1}{2}\right) = \frac{3x + 19}{2} - \left(\frac{x + 1}{6} + 3\right).$$

4. Reduce to lowest terms:

(a) $\frac{32}{24}$. (b) $\frac{a^6 - x^6}{a^4 - x^4}$.

5. If $a = 3$, $b = 4$, $c = 2$, $d = 6$, $m = 8$, and $n = 10$, find the value of the following expressions:

(a) $\frac{ad}{c} + a + mn$. (c) $\frac{b + ad}{c} + bcm - \frac{4cdn}{5a}$.

(b) $\frac{b + mn}{cd} + \frac{bc + n}{3d}$.

6. Find the product of $2x - 7y + 3z$ and $2x + 7y - 3z$.

7. Divide $(x^2 + 7x + 10)(x + 3)$ by $x^2 + 5x + 6$.

8. Prove that $(x + y)^3 - (x - y)^2(x + y) = 4xy(x + y)$.

9. Factor by the factor theorem:

(a) $x^3 + 3x^2 - 13x - 15$. (b) $x^3 + 6x^2 + 11x + 6$.

10. $\dfrac{x^2 - x - 20}{x^2 - 25} \cdot \dfrac{x^2 - x - 2}{x^2 + 2x - 8} \div \dfrac{x + 1}{x^2 + 5x} = ?$

11. Add $\dfrac{2x}{4 - x^2} + \dfrac{1}{2 - x} - \dfrac{1}{2 + x}$.

12. Express each of the following fractions as a group of simple fractions in lowest terms:

(a) $\dfrac{yz + zx + xy}{xyz}$. (b) $\dfrac{x + y + z}{xyz}$.

13. Factor:

(a) $x^4 - y^4$.
(b) $(x + y)^2 - 81$.
(c) $x^2 - 9x - 52$.
(d) $x^2 + 17xy + 60y^2$.
(e) $10x^2 + 79x - 8$.
(f) $x^3 - 2x^2 + 1$.

14. Simplify:

$$2a - [5d + \{3c - (a + [2d - 3a + 4c])\}].$$

15. Write the product of

(a) $(.2x + 1.2y)^2$.
(b) $(2x + 3y)(3x - .2y)$.
(c) $(x^{n+1} - y^m)^2$.
(d) $(x^2 + x + 1)^2$.

16. Find a number such that if four fifths of the number be added to 6 it will give the same result as one sixth of the number added to 25.

17. $4\frac{1}{6} \div \dfrac{4x - 5y}{6x - 6y} = ?$

18. Solve $x - \left(\dfrac{1 - x}{4} - \dfrac{1 + x}{2}\right) = \dfrac{x + 3}{2}$.

19. How can you check the solution of an equation?

20. Find the least expression into which each of the following will divide with a remainder of K.

$$x^2 - 25, \qquad x^2 - 10x + 25.$$

21. The square of a certain number is three times the number. Find the number.

22. Show that

$$(x + y)(x + y - 1) = x(x - 1) + 2xy + y(y - 1).$$

23. Find the value of

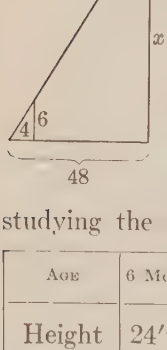
$$[a - (b - c)]^2 + [b - (c - a)]^2 + [c - (a - b)]^2.$$

24. Simplify:

$$\frac{\dfrac{x^2 + y^2}{x} - y}{\dfrac{1}{x} - \dfrac{1}{y}} \div \frac{x^3 + y^3}{x^3 - y^3}.$$

25. If a pole casts a shadow on the level ground of 48 feet, when a pole 6 feet high casts a shadow of 4 feet on the level ground we have the following fractional relation.

$$\frac{x}{48} = \frac{6}{4}.$$



(a) Find the height of the pole.

(b) Put 56 feet in the place of the 48 feet and find the height of the pole.

26. The following average heights of children at different ages were obtained by studying the heights of many children.

Age	6 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.	7 Yr.	8 Yr.
Height	24″	29″	36″	40″	42″	44″	46″	48″	50″

Hint: 24″ means 24 inches.

Make a graphical representation of the above facts.

27. By how much does $x^3 - 3x^2 + 8x + 9$ exceed $x - 4x^2 + 6 - x^3$?

28. Solve and check:

$$\tfrac{1}{3}(x + 1) + \tfrac{1}{4}(x + 3) = \tfrac{1}{5}(x + 4) + 16.$$

29. Factor:

(a) $x^3 + 27y^3$.

(b) $x + hx - kx + y + hy - ky$.

(c) $x^3 - 6x^2 + 11x - 6$.

(d) $16x^5 - x$.

30. Solve (a) $2r - h = \dfrac{w^2}{4h}$ for the value of r.

(b) Find the value of r when $w = 16$ ft. and $h = 4$ ft.

31. Write as a single fraction

$$\frac{1}{x + 3} + \frac{7}{x^2 - x - 12} + \frac{x + 3}{x - 4}.$$

32. A certain college decided to allow all high school graduates who were above the median of the standings of all of the graduates of a certain school to enter their institution without taking entrance examinations. The four-year standings of the graduates in this school were 81.4, 89.7, 75, 94.2, 79.5, 87.5, 96.3, 77.6, 92.1, 80.2, and 84.3.

(a) Arrange these standings in proper order and find the median mark.

(b) How many of these pupils could enter that college?

(c) What can you say in favor of this system? Against it?

33. What extraneous root is introduced into the equation

$$4x - 6 = 2x + 4$$

by multiplying each member by $x - 4$?

34. Find the least common multiple of the following expressions:

$$a^4 + a^2x^2 + x^4, \quad a^2 + ax + x^2, \quad a^2 - ax + x^2.$$

35. Given $x = 5$ and $y = 2$, evaluate the following expressions:

(a) $(x + 2y)(x - 2y)$.

(b) $x + 2y(x - 2y)$.

(c) $x + 2yx - 2y$.

(d) $\dfrac{x + 2y}{x - 2y}$.

(e) $x + \dfrac{2y}{x - 2y}$.

(f) $\dfrac{x}{x - 2y} + 2y$.

36. Solve and check: $2.3(x - 3) - .4(3x + 2) = 2.2$.

37. The denominator of a fraction is 6 more than the numerator. What is the fraction if its value equals .25?

38. What do each of the following fractions mean?

$$\frac{x}{x+6},\ \frac{x-4}{x},\ \frac{x}{x-5},\ \frac{x+4}{x},\ \frac{x+\frac{1}{2}}{x},\ \frac{\frac{1}{2}x}{x}.$$

What is the value of the last fraction?

39. The denominator of a fraction is three times the numerator. If the numerator is increased by 4, and the denominator is increased by 2, the value of the fraction is three fourths. What was the original fraction?

40. The second term of a ratio is 12 more than the first term. If 6 is added to both terms, the value of the ratio is $\frac{3}{7}$. What were the terms of the original ratio? What was that ratio?

41. (a) What is the ratio of the number of games of baseball won to the number played in each of the following: Won 50, lost 25; won 80, lost 20; won x, lost $x + 20$; won $x + 10$, lost $x + 30$?

(b) State the per cent won in each of the above.

42. If a baseball team wins 30 games more than it loses during the season and the percentage of games won at the end of the season is 80, how many games were lost? How many won? How many played?

43. A tennis player in a round-robin tournament (every player plays every other player) has won 6 games out of the 10 he has played. If he finishes the tournament with a percentage of 80, how many more games must he play? How many did he win in all?

44. How many quarts of pure water must be added to one gallon of alcohol which is 75% pure to produce a mixture 60% pure?

Hint: One fourth of the gallon was water at first.

45. What will be the resulting temperature if 40 pounds of water at 160° are mixed with 30 pounds of water at 40°?

Hint: The former loses $40(160 - x)$ heat units.
The latter gains $30(x - 40)$ " "
These are equal.

46. What is the resulting temperature if 50 pounds at 36° are mixed with 80 pounds at 200°?

47. Prove that $\dfrac{a^{6m} + a^{3m}}{a^{2m} + a^m} = a^{4m} - a^{3m} + a^{2m}$.

Check when $a^{4m} = 16$.

48. Simplify $\dfrac{x^3 + x^2y}{x^2y - y^3} - \dfrac{x(x - y)}{xy + x^2} - \dfrac{2\,xy}{x^2 - y^2}$.

49. Find whether $x =$ one third is a root of

$$\frac{x + \frac{1}{5}}{\frac{8}{15}} + \frac{x + \frac{1}{6}}{\frac{1}{2}} - \frac{3(x - \frac{1}{3})}{\frac{1}{16}} = \frac{2(x + \frac{1}{3})}{\frac{2}{3}}.$$

50. A man sells $\frac{1}{2}$ a pound more than one half of all the pounds of sugar he had. He next sold $\frac{1}{2}$ a pound more than one half of the pounds that were not yet sold, and then he had 6 pounds left. How many had he at first?

51. Find the value of

$$(x - y)[(x + z) + (y - z)] - x^2 + y(y + z) - zy$$

when $x = 1$, $y = 2$ and $z = 3$.

52. Is $x - 3$ a factor of $x^3 - 2x^2 - 5x + 6$? Prove.

53. What is the continued product of $x^m - a^n$, $x^m + a^n$, and $x^{2m} - a^{2n}$?

54. Find the prime factors of

(a) $14x^2 - 11x - 15$.

(b) $x^4 + 5x^2y^2 + 9y^4$.

(c) $4x^2 - 4xy - 15y^2$.

(d) $9x^4 - 82x^2y^2 + 9y^4$.

What is the result of

55. (a) Dividing the ratio $4:5$ by x?

(b) " " " $1:x$ " 25?

(c) " " " $m:n$ " y?

56. Add: $3 \cdot 2^2$, $4 \cdot 2^2$, $6 \cdot 2^2$, $8 \cdot 2^2$.

57. Multiply $x^{a+1} + y^b$ by $x^{2a+2} - x^{a+1}y^b + y^{2b}$.

58. Solve and check:

$$4(x - 6) - 2[3x - (x - 8)] = 5(13 - 3x).$$

59. The width of a floor is 2 feet less than the length. If each dimension is increased 2 feet, the area of the floor will be increased by 48 square feet. Find the dimensions of the floor.

60. Find the value of $1 + \dfrac{1}{1 + \dfrac{1}{x}}$ when $x = 3$.

CHAPTER V

POWERS AND ROOTS

EARLIER in our course we learned how to add, subtract, multiply, and divide algebraic quantities. Two other processes are often necessary, especially in the application of algebra to geometry. These are the processes of raising to powers and of extracting roots. The present chapter will show how these operations are performed and will give some of their applications.

59. Involution is the process of raising a quantity to an indicated power.

We worked many examples of this type under the head of multiplication, of which it is a special case.

Illustrative examples:

I. $(4x^2)^2 = 4x^2 \cdot 4x^2 = 16x^4$.

II. $(2x^2y^4)^3 = 2x^2y^4 \cdot 2x^2y^4 \cdot 2x^2y^4 = 8x^6y^{12}$.

III. $(-3x^5)^2 = (-3x^5)(-3x^5) = +9x^{10}$.

IV. $\left(\frac{-2a^2}{3bx^3}\right)^3 = -\frac{8a^6}{27b^3x^9}$.

By studying these illustrations we may infer the following

Rule. — *To raise an algebraic quantity to a given power,* (1) *raise the numerical coefficient to the required power and* (2) *multiply each of the exponents of the given quantity by the exponent of the power to which it is to be raised.*

The law of exponents in involution may be expressed in symbols thus $(x^m)^n = x^{mn}$.

NOTE. You should notice that an even power of any quantity is always positive, and an odd power of any quantity has the same sign as the given quantity.

Oral Exercises

Perform the indicated operations in each case:

1. $(3x)^2$.
2. $(-3x)^2$.
3. $(4y^2z)^2$.
4. $(-4y^2z)^2$.
5. $(6x^3)^2$.
6. $(-6x^3)^2$.
7. $(3mn^3y)^2$.
8. $(-3mn^3y)^2$.
9. $(5ab^4)^2$.
10. $(-5ab^4)^2$.
11. $(12x^5)^2$.
12. $(-12x^5)^2$.
13. $(13xyz)^2$.
14. $(-13xyz)^2$.
15. $(25a^2b^2)^2$.
16. $(-25a^2b^2)^2$.
17. $(15HK)^2$.
18. $(-15HK)^2$.
19. $(4x^2)^3$.
20. $(3ab^2)^3$.
21. $(-2x)^3$.
22. $(-3m^2n)^3$.
23. $(5m^3x^2)^3$.
24. $(-2x^2yz)^3$.
25. $(3x^2)^4$.
26. $(-2ms)^4$.
27. $(4x)^4$.
28. $(2x^2)^5$.
29. $(-y^2)^5$.
30. $(-2x^3y)^5$.
31. $(\frac{1}{2}a^2b)^2$.
32. $(3^2)^2$.
33. $(\frac{1}{2}\cdot\frac{1}{3})^2$.
34. $\left(\dfrac{-4x^2y^3}{5ab^2}\right)^2$.
35. $((a-b)^2)^2$.
36. $((-3)^2)^2$.
37. $((+3)^2)^2$.
38. $((-2)^2)^3$.
39. $(x^3)^2\cdot x^5$.
40. Is $((-2)^2)^3$ equal to $((-2)^3)^2$?

60. Evolution is the process of extracting a root.

The square root of a quantity is one of the two equal factors of that quantity. The cube root of a quantity is one of the three equal factors of the quantity. A similar definition holds for higher roots.

Since $(3x)^2 = 9x^2$ and $(-3x)^2$ also equals $9x^2$, it follows from the above definition that both $3x$ and $-3x$ are square roots of $9x^2$. We may therefore state the following

Principle. — *Every algebraic number or expression has two square roots, numerically equal, but opposite in sign.*

Since $(2x)^3 = 8x^3$ and $(-2x)^3 = -8x^3$ we see that the cube root of $8x^3$ is $2x$ and the cube root of $-8x^3$ is $-2x$.

A quantity, therefore, has but one real cube root, and that has the same sign as the quantity.

61. Radical Signs. — The extraction of a root is often indicated by the sign $\sqrt{\ }$, called the *radical sign*. Thus, $\sqrt{16x^4y^6}$ is read "the square root of $16x^4y^6$" and means that we are to find the square root of that quantity. Similarly, $\sqrt[3]{8x^6}$ means "the cube root of $8x^6$" and $\sqrt[4]{81x^{12}}$ means "the fourth root of $81x^{12}$." The small figure at the upper left-hand corner of the radical sign is called the *index* of the root, and shows what kind of root is to be taken. In the square root sign the index 2 is implied.

62. Laws of Signs and Exponents in Evolution. — The principle governing signs in the extraction of square root, which was explained in section 60, holds for any root whose index is an even number, and the principle explained for cube root holds for any root whose index is an odd number.

Exponents in the extraction of roots may be found by the law $\sqrt[n]{x^m} = x^{\frac{m}{n}}$, that is, we divide the exponent by the index of the root. Thus $\sqrt{x^6} = \pm x^3$. $\sqrt[3]{x^{12}y^{18}} = x^4y^6$, etc.

We now state the following

Rule. — *To extract the root of a monomial*

(1) *Find by trial the indicated root of the numerical coefficient.*

(2) *Annex each literal factor with an exponent equal to that found in the given expression divided by the index of the root.*

(3) *Prefix the* $\pm$ *sign if the index is even. Prefix the given sign if the index is odd.*

The sign $\pm$ is read plus or minus.

Illustrative examples:

I. The square root of $4\,x^4y^{10}$ is $\pm\,2\,x^2y^5$.

II. The cube root of $-\,27\,x^6y^9$ is $-\,3\,x^2y^3$.

III. The fourth root of $16\,x^8y^{20}$ is $\pm\,2\,x^2y^5$.

IV. The fifth root of $\frac{1}{32}\,a^5b^{15}$ is $+\,\frac{1}{2}\,ab^3$.

Oral Exercises

What is the square root of each of the following:

1. $64\,x^8$.
2. $36\,m^2n^6$.
3. $16\,a^4b^{10}$.
4. $144\,a^8b^4$.
5. $9\,m^2n^2x^6$.
6. $25\,a^8b^{12}$.
7. $81\,x^{16}z^{20}$.
8. $121\,a^{10}m^4n^2$.
9. $49\,x^{14}y^{12}z^{20}$.
10. $100\,s^2t^4$.
11. $25(a-b)^2s^6$.
12. $\frac{1}{4}(x-2\,y)^4$.
13. $\frac{1}{16}\,s^2t^6$.
14. $\frac{4}{9}\,a^4b^6$.
15. $\frac{9}{16}\,a^8b^2$.
16. $\frac{25}{81}\,m^{10}n^{16}$.
17. $.09\,x^4y^2$.
18. $.16\,a^8b^4$.
19. $(H+K)^2(M-N)^4$.
20. $64\,x^2s^{10}(a-b)^2$.
21. $36\,x^{20}y^{12}$.
22. $\dfrac{(x+y)^2}{4}$.
23. $\dfrac{(x-y)^2}{(x+y)^2}$.
24. $.25\,x^8y^6$.

25. The sum of one eighth and one eighth.

26. The result of subtracting three sixteenths from four sixteenths.

27. The product of one half and one eighteenth.

28. 2 times 18. **29.** $9\,x^{8a}$. **30.** $.16\,x^{2a}$.

31. What value of x makes $x^2 = 25$?

32. What value of x makes $x^4 = 16$?

In examples 33–39, use only positive square roots.

33. $\sqrt{25} + 5 + \sqrt{4} = ?$ **36.** $\dfrac{2\sqrt{81} - \sqrt{16}}{7} = ?$

34. $\sqrt{4 \cdot 25} = ?$

35. $\sqrt{4 + 21} + \sqrt{16} = ?$

37. What is the value of $\sqrt{x} - \sqrt{y}$ if $x = 25$ and $y = 16$?

38. What is the value of $4\sqrt{x} + 3\sqrt{y}$ if $x = 16$ and $y = 9$?

39. (a) $\sqrt{16 + 9} = ?$ (b) $\sqrt{16} + \sqrt{9} = ?$

(c) How do these two results compare?

40. What is the square root of $\dfrac{.25\,x^2}{.36\,y^4}$?

State the cube root of each of the following:

41. $27\,x^9y^{18}$. **43.** $-8\,a^3b^{15}$.

42. $64\,m^3n^{12}$. **44.** $125\,m^{15}n^{30}$.

45. $-27\,x^{21}(a - b)^3$.

What is the fourth root of

46. $16\,x^4$? **49.** $(a - b)^8(x + y)^{12}$?

47. $81\,y^8x^{12}$? **50.** $\dfrac{a^4b^{16}}{16}$?

48. $256\,m^{20}$?

State the fifth root of

51. $-32\,x^{10}y^{15}$. **52.** $243(a - b)^5c^{15}$. **53.** $.00032\,x^{15}(a + b)^{30}$?

63. Square Root of a Trinomial. — A rational binomial expression is never a perfect square, and its square root can, therefore, not be found exactly.

A trinomial square is, however, a very common algebraic expression. Such a square is formed by squaring a binomial, as explained in section 32. By the rule there stated $(3x - 4y)^2 = 9x^2 - 24xy + 16y^2$. The square root of $9x^2 - 24xy + 16y^2$ is, therefore, $3x - 4y$. $-(3x - 4y)$ is also a square root of $9x^2 - 24xy + 16y^2$, so that we ought to inclose $3x - 4y$ in a parenthesis preceded by a $\pm$ sign; thus, $\pm(3x - 4y)$.

The square root of a trinomial square like the above may always be found mentally by the following

Rule. — (1) *Take the square root of the first term of the trinomial and the square root of the last term.*

(2) *Make the first of these terms positive, and give the second the sign of the middle term of the trinomial.*

(3) *Inclose the resulting binomial in a parenthesis preceded by the* $\pm$ *sign.*

In applying the above rule the trinomial must first be arranged in either the descending or ascending powers of some common letter.

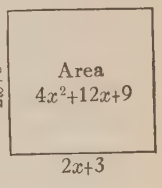
Oral Exercises

State the square roots of each of the following trinomials:

1. $x^2 + 6x + 9$.
2. $x^2 - 12x + 36$.
3. $x^2 - 8xy + 16y^2$.
4. $4x^2 - 12xy + 9y^2$.
5. $25x^2 - 10x + 1$.
6. $25a^2 + 30ab + 9b^2$.
7. $100m^2 - 40mn + 4n^2$.
8. $36x^2 + 12xy + y^2$.
9. $x^2 - 14xz + 49z^2$.
10. $16x^2 - 16x + 4$.
11. $s^2 - 20s + 100$.
12. $81 + 54x + 9x^2$.
13. $16x^2 + 8 + \frac{1}{x^2}$.
14. $a^2 + \frac{4a}{b} + \frac{4}{b^2}$.
15. $.04x^2 + .12xy + .09y^2$.

64. Square Roots of Polynomials. — We shall now explain the process of extracting the square root of polynomials in general.

We may find the area of a square, when one side is known, by squaring that side. For example, the square whose side is $2x + 3$ has an area equal to $(2x + 3)^2$, or $4x^2 + 12x + 9$. If, however, the area of the square were given, we should reverse the process and extract the square root of the area to obtain the length of the side.



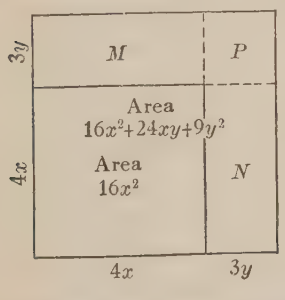
For example, the square whose area is 64 square feet has a side whose length is $\sqrt{64}$, or 8 feet.

Why is the negative square root in this case an unsatisfactory answer?

Illustrative example I:

Suppose the area of a square is $16x^2 + 24xy + 9y^2$. We may find the side by the following process.

Solution: (1) We note that the perfect square $16x^2$ forms a part of this area. It represents a square whose side is $4x$. We subtract this square from the given square and we have left an irregular rectangular figure $(M + P + N)$ whose area must be the remainder of the square, that is, $24xy + 9y^2$.



(2) Now we wish to obtain the width of this rectangular portion, because that is what is left of the given square. This we may do by dividing the area, which we know, by the length. But we do not know the length, so we must approximate it by a trial divisor.

For this trial divisor we take $2 \cdot 4x = 8x$ as the approximate length, for we know that the length of each of the rectangles M and N is $4x$. We must remember to add the length P later when we find it. All we know of P is that its length is the same as its width.

(3) Now, dividing the first term of the area $24xy + 9y^2$ by $8x$, we obtain as the probable width $3y$. Then $3y$ is also the length of the square P.

$$\begin{array}{r|l} & 16x^2 + 24xy + 9y^2 \\ & 16x^2 \\ \hline 8x + 3y & 24xy + 9y^2 \\ & \underline{24xy + 9y^2} \end{array}$$

$\therefore \pm (4x + 3y)$ is the answer.

The total length of the rectangular figure is, therefore, $8x + 3y$. Multiplying this length by the assumed width $3y$, we obtain $24xy + 9y^2$, which exactly agrees with the given area.

The side of the square is, therefore, $4x + 3y$. This process may be extended indefinitely.

We now state the following rule for extracting the square root of a polynomial.

Rule. — (1) *Arrange the terms of the polynomial in descending or ascending powers of a common letter.*

(2) *Extract the square root of the first term of the polynomial to obtain the first term of the root.*

(3) *Square this term and subtract its square from the polynomial.*

(4) *Multiply the first term of the root by 2 to obtain the trial divisor.*

(5) *Divide this trial divisor into the first term of the remainder to obtain the second term of the root.*

(6) *Annex this term also to the trial divisor to obtain the complete divisor.*

(7) *Multiply the complete divisor by the second term of the root, subtract the product from the remainder, and so continue*

until the root is obtained exactly, or to as many terms as may be required.

(8) *Inclose in parenthesis, preceded by the* $\pm$ *sign.*

Illustrative example II:

We shall now illustrate the above rule by extracting the square root of $9x^4 - 12x^3 + 34x^2 - 20x + 25$.

This polynomial is already arranged in proper order.

Solution:

The square root of $9x^4$ is $3x^2$. $\quad 9x^4 - 12x^3 + 34x^2 - 20x + 25 \mid 3x^2 - 2x + 5$

$(3x^2)^2 = 9x^4$

Subtract $\quad -12x^3 + 34x^2 - 20x + 25$

Trial divisor $= 2 \cdot 3x^2 = 6x^2$

$-12x^3 \div 6x^2 = -2x$

Complete divisor $= 6x^2 - 2x$

$-2x(6x^2 - 2x) = -12x^3 + 4x^2$

Subtract $\quad 30x^2 - 20x + 25$

Trial divisor $= 2(3x^2 - 2x) = 6x^2 - 4x$

$30x^2 \div 6x^2 = 5$

Complete divisor $= 6x^2 - 4x + 5$

$+5(6x^2 - 4x + 5) = 30x^2 - 20x + 25$

The required root is therefore $\pm (3x^2 - 2x + 5)$.

Check: This root if squared should produce the original expression.

Illustrative example III:

Find the square root of $\frac{x^2}{y^4} + \frac{4x}{y^3} + \frac{10}{y^2} + \frac{12}{xy} + \frac{9}{x^2}$.

Solution: The given arrangement is in descending powers of x.

$$\left.\begin{array}{l} \text{The square root of } \dfrac{x^2}{y^4} \text{ is } \dfrac{x}{y^2}. \\ \left(\dfrac{x}{y^2}\right)^2 = \dfrac{x^2}{y^4} \end{array}\right| \begin{array}{l} \dfrac{x}{y^2} + \dfrac{2}{y} + \dfrac{3}{x} \\ \hline \dfrac{x^2}{y^4} + \dfrac{4\,x}{y^3} + \dfrac{10}{y^2} + \dfrac{12}{xy} + \dfrac{9}{x^2} \\ \hline \end{array}$$

$$2\left(\frac{x}{y^2}\right) = \frac{2\,x}{y^2};\ \frac{4\,x}{y^3} \div \frac{2\,x}{y^2} = \frac{2}{y}. \qquad \frac{4\,x}{y^3} + \frac{10}{y^2} + \frac{12}{xy} + \frac{9}{x^2}$$

Complete divisor is

$$\frac{2\,x}{y^2} + \frac{2}{y};\ \frac{2}{y}\left(\frac{2\,x}{y^2} + \frac{2}{y}\right) = \qquad \frac{4\,x}{y^3} + \frac{4}{y^2}$$

$$2\left(\frac{x}{y^2} + \frac{2}{y}\right) = \frac{2\,x}{y^2} + \frac{4}{y};\ \frac{6}{y^2} \div \frac{2\,x}{y^2} = \frac{3}{x}. \qquad \frac{6}{y^2} + \frac{12}{xy} + \frac{9}{x^2}$$

Complete divisor =

$$\frac{2\,x}{y^2} + \frac{4}{y} + \frac{3}{x};\ \frac{3}{x}\left(\frac{2\,x}{y^2} + \frac{4}{y} + \frac{3}{x}\right) = \qquad \frac{6}{y^2} + \frac{12}{xy} + \frac{9}{x^2}$$

We have, therefore, $\pm\left(\dfrac{x^2}{y^2} + \dfrac{2}{y} + \dfrac{3}{x}\right)$ as the root.

Exercises

Extract the square root of each of the following polynomials:

1. $9\,x^4 + 24\,x^3 + 10\,x^2 - 8\,x + 1$.
2. $25\,x^4 - 10\,x^3 + 61\,x^2 - 12\,x + 36$.
3. $9\,n^4 - 12\,n^3 - 38\,n^2 + 28\,n + 49$.
4. $16\,m^6 - 24\,m^4 - 16\,m^3 + 9\,m^2 + 12\,m + 4$.
5. $36\,n^6 - 60\,n^5 + 25\,n^4 + 36\,n^3 - 30\,n^2 + 9$.
6. $16\,x^6 - 16\,x^5 - 4\,x^4 + 28\,x^3 - 11\,x^2 - 6\,x + 9$.
7. $9\,a^6 + 12\,a^5 - 26\,a^4 - 8\,a^3 + 33\,a^2 - 20\,a + 4$.
8. $4\,y^4 - 12\,y^3 - 7\,y^2 + 24\,y + 16$. Check.

9. $x^4 - 2x^3y + 5x^2y^2 - 4xy^3 + 4y^4$.

10. $9s^4 - 42s^3t + 61s^2t^2 - 28st^3 + 4t^4$.

11. $36a^4 - 36a^3b - 15a^2b^2 + 12ab^3 + 4b^4$.

12. $64a^4 + 16a^3b + 17a^2b^2 + 2ab^3 + b^4$.

13. $\frac{x^2}{4} + 9 - 3x$.

14. $\frac{4}{25}x^2 + \frac{3}{5}xy + \frac{9}{16}y^2$.

15. $\frac{1}{4}a^4 + \frac{1}{3}a^3 + \frac{19}{9}a^2 + \frac{4}{3}a + 4$.

16. $\frac{4}{9}x^4 - \frac{1}{3}x^3 + \frac{143}{240}x^2 - \frac{1}{5}x + \frac{4}{25}$.

17. $\frac{1}{4}s^4 - \frac{3}{5}s^3 + \frac{147}{200}s^2 - \frac{9}{20}s + \frac{9}{64}$.

Extract the fourth root of the following by extracting the square root, and then the square root of that result:

18. $x^4 + 8x^3 + 24x^2 + 32x + 16$.

19. $x^4 - 12x^3 + 54x^2 - 108x + 81$.

Extract the square root of:

20. $\frac{x^2}{y^2} + \frac{6x}{y} + 11 + \frac{6y}{x} + \frac{y^2}{x^2}$.

21. $\frac{a^2}{b^2} + 3 - \frac{2b}{a} - \frac{2a}{b} + \frac{b^2}{a^2}$.

22. $\frac{1}{m} + \frac{m^4}{4} + \frac{x^2}{m^2} + mx + \frac{m^2}{2x} + \frac{1}{4x^2}$.

23. $\frac{x^4}{9} + \frac{2x^2}{3} + 1 + \frac{4}{x^2} - \frac{4}{x} - \frac{4x}{3}$.

24. $\frac{a^2}{x^2} + \frac{x^2}{a^2} + \frac{1}{4} - \frac{a}{x} - \frac{x}{a} + 2$.

25. $.09x^2 + .24x + .16$.

26. $.04x^2 + .12xy + .09y^2$.

27. $1.44x^2 - .72x + .09$.

28. $.04x^2 + .12xy - .4x + .09y^2 - .6y + 1$.

29. $1 + x^2 + 4y^2 + 2x - 4y - 4xy$.

30. $.09\,x^2 + .12\,x - .24\,xy + 4 - 1.6\,y + .16\,y^2$.

31. $x^2 - 2$ to three terms.

Hint: This means work this out only until you have three terms in the root.

32. $9 - x$ to three terms.

65. Square Roots of Numbers. — The process of extracting square roots of numbers is familiar to you. It is desirable, however, to review this topic at this point and to notice the similarity between the method and that of extracting square roots of polynomials.

Illustrative example:

Let it be required to extract the square root of 138,384. The number must first be pointed off in groups of two figures each, beginning at the decimal point, which, in the case of an integer, is at the right. The number of figures in the root agrees with the number of groups so obtained.

Solution: The largest square less than 13 is 9. $\sqrt{9} = 3$.

Subtract, bringing down one group

Annex 0 to 3 (making 30) and multiply by 2. Trial divisor $= 2 \cdot 30 = 60$. $483 \div 60 = 8$. However, as the trial divisor must later be increased, 8 will be found too large. We therefore take 7 as the second figure. Complete divisor $= 60 + 7 = 67$

Subtract, bringing down a group

Trial divisor $= 2 \cdot 370 = 740$

$1484 \div 740 = 2$

Complete divisor $= 740 + 2 =$

$$\begin{array}{rl} & 3\ 7\ 2 \\ & \overline{13'83'84} \\ 3^2 = & 9 \\ & \overline{4\ 83} \\ 7 \cdot 67 = & 469 \\ & \overline{14\ 84} \\ & 742 \\ 2 \cdot 742 = & \underline{14\ 84} \end{array}$$

Exercises

In like manner extract the square root of:

1. 7396. 2. 15,129. 3. 183,184. 4. 813,604.

Find to two decimal places the square root of

5. 328.66.

Solution:

$$
\begin{array}{rr}
 & 1\ 8.\ 1\ 2 \\
 & \overline{3'28.66'00} \\
 & 1 \\
\hline
2 \cdot 10 = 20;\ 20 + 8 = 28 = & 2\ 28 \\
8 \cdot 28 = & 2\ 24 \\
\hline
2 \cdot 180 = 360;\ 360 + 1 = 361 & 4\ 66 \\
1 \cdot 361 = & 3\ 61 \\
\hline
2 \cdot 1810 = 3620 & 1\ 05\ 00 \\
3620 + 2 = 3622;\ 2 \cdot 3622 = & 76\ 44 \\
\hline
\end{array}
$$

$\therefore$ The square root of 328.66 is $\pm$ 18.12+.

NOTE. Had we asked for the root correct to the nearest hundredth, we should have carried this out to three decimals, obtaining 18.127 or 18.13.

6. 3.0625. 9. .3856. 12. 296,172.

7. 458.49. 10. .00168. 13. 25.06.

8. 16.9744. 11. 4287. 14. 286.33.

15. $\frac{3}{8}$. *Hint:* First change the fraction to a decimal.

16. $\frac{5}{16}$. 17. $3\frac{5}{9}$. 18. $8\frac{2}{3}$. 19. $12\frac{5}{8}$.

Find to the *nearest hundredth* the square root of:

20. 394.1728. 22. 1194.6. 24. $22\frac{1}{3}$.

21. 16.72. 23. $\frac{5}{11}$. 25. 18.0625.

Find to three decimals the square root of:

26. (a) 3. (b) 2.

Memorize both your answers.

27. Find to the nearest tenth the number of rods in the side of a square field which contains 8 acres.

Hint: 1 acre = 160 square rods.

28. The area of a triangle whose sides are represented by a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

Find the area of a triangle whose sides are 24, 18, and 16 inches, respectively.

Exercises

In the right triangle the square of the longest side equals the sum of the squares of the other two sides.

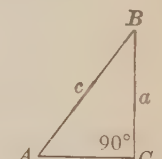
Thus in the adjoining figure

$$a^2 + b^2 = c^2.$$

Hence $c = \sqrt{a^2 + b^2}$.

Also $a = \sqrt{c^2 - b^2}$,

and $b = \sqrt{c^2 - a^2}$.



1. Find the value of c in a right triangle whose altitude (a) = 8, and whose base (b) = 6.

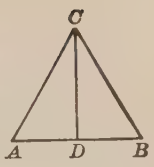
2. Solve the equation $c^2 = a^2 + b^2$ for a^2. For a. Then find the value of a if $c = 5$ and $b = 4$.

3. Solve the equation $c^2 = a^2 + b^2$ for b^2. For b. Then find the value of b if $c = 5$ and $a = 3$.

4. A ladder 25 feet long is even with the top of a wall a feet high. The foot of the ladder is 13.5 feet away from the foot of the wall. How high is the wall?

5. A railway incline is 5 in each 150 (*i.e.* 5 feet high for each 150 feet of slant height, or when $c = 150$, $a = 5$). Find the projected or horizontal length.

6. A perpendicular is drawn from the vertex to the base of an equilateral (all sides equal) triangle each of whose sides is 4 inches. Find the length of the perpendicular CD.



7. The side of a square is 2 inches. Find the length of the straight line which joins two opposite corners of the square.

8. If it is 90 feet from home base to first base on a baseball diamond, and 90 feet from first base to second base, find the shortest distance from home base to second base.

9. Find the hypotenuse and the area of a right triangle whose legs are 42 feet and 26 feet, respectively.

10. Find the base of a right triangle whose hypotenuse (c) = 33 feet, and whose altitude (a) = 28 feet. Find the area.

11. Find the diagonal (the line joining the opposite corners) of a rectangle, if the base is 70 feet and the altitude (a) is 12 feet.

12. The diagonal (c) of a certain rectangle is twice the length of the altitude (a) of the rectangle. If the base (b) of that rectangle is 5 inches, find the altitude and the diagonal. Find the area of the rectangle.

13. The velocity of the flow of water under a head is denoted by $v = c\sqrt{2gh}$. Find v when

$$c = .62$$
$$g = 32.$$
$$h = 15.7.$$

14. The velocity of a falling body is denoted by the formula $v = \sqrt{2gs}$. What is the velocity of an object which falls from a height (s) of 555 feet (the height of the Washington monument in Washington) when it strikes the ground?

Hint: $g = 32.16$.

15. Find to the nearest tenth of an inch the side of a square whose area is 600 square inches.

16. Find to the nearest tenth of a foot the side of a square whose area is equal to the area of a rectangle with a base of 50 feet and an altitude of 20 feet.

17. The altitude of a right triangle is $1\frac{1}{3}$ times the base. The hypotenuse is 15. Find the base and the altitude.

Hint: If b = base, a = altitude, c = hypotenuse, $a^2+b^2=c^2$.

18. In a right triangle the longer leg is 2 inches more than the shorter leg and 2 inches less than the hypotenuse. Find each of the sides of the triangle.

19. One leg of a right triangle is 3 feet longer than the other leg, and the area of the triangle is 5 square feet. Find the length of each leg.

20. The area of a circle is πr^2. What is the radius of a circle whose area is 37.65 square feet?

21. Find to the nearest hundredth the value of $\sqrt{2+\sqrt{3}}$. Of $\sqrt{3\sqrt{6}-\sqrt{5}}$.

22. If a person were to fly a kite out 450 feet and another person noted that the kite was directly above a building 360 feet away from the one who flew the kite and on the same level, how high would the kite be?

23. Solve $x^2 = 5 + \sqrt{6}$ for the value of x.

24. Find the altitude of an equilateral triangle whose side is 10.

Hint: See example 14, page 252, First Course.

25. Find the altitude of an equilateral triangle whose side is s.

26. Find the side of an equilateral triangle whose area is 50.

27. What is the area of a regular hexagon whose side is a?

Hint: The hexagon may be divided into six equilateral triangles.

28. What is the area of a regular hexagon whose side is 5?

29. What is the radius of a circle whose area is equal to the area of a rectangle 9 inches by 5 inches? What is the circumference of that circle?

CHAPTER VI

EXPONENTS

Up to this time you have learned to use exponents quite accurately, but you have used only exponents which were positive integers.

Frequently we meet with exponents of different kinds, and in this chapter we shall explain their meaning and use.

66. Laws of Exponents. — In earlier sections of this book you will find stated the following laws of exponents.

For multiplication $a^m \cdot a^n = a^{m+n}$.

For division $\frac{a^m}{a^n} = a^{m-n}$.

For involution $(a^m)^n = a^{mn}$.

For evolution $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

The exponents to which they applied, however, were always positive integers, and they were applied only when the results of their application were also positive integral exponents. In fact, up to this time we have known no other kind of exponent.

What is the definition of an exponent which you learned in your first course in algebra?

State the exponent in each of the following:

(a) $x^2 = 9$. (b) $x^6 = 8$. (c) $x^4 = 16$. (d) $x^5 = 32$. (e) $x^6 = 64$.

State the value of x in each of the above.

Would your definition of an exponent have a meaning if the exponent were zero, negative, or fractional?

Such exponents do occur, inevitably, as a result of applying the above laws to all cases, and their meanings will be explained in the sections which follow.

67. The Zero Exponent. — We now wish to extend the meaning of powers by assuming that

$$\frac{a^m}{a^n} = a^{m-n}$$

is also true when $m = n$.

We now have by the law of exponents in division

$$\frac{a^n}{a^n} = a^{n-n} = a^0.$$

But if we divide numerator by denominator, $\frac{a^n}{a^n} = 1$.

Therefore $a^0 = 1$, since both are equal to the same thing.

Value of a^0.

Hence: *Any quantity (except 0) with a zero exponent has a numerical value of 1.*

Thus $x^0 = 1$, $5^0 = 1$, and $[3\,x(2\,a - b)^3]^0 = 1$. $5\,a^0 = 5$.

We have, therefore, shown that the stated law holds if $m = n$, and we obtain a definite value of any number (except 0) to the zero power.

Meaning of a^0.

You should notice that a^0 is merely a symbol for $\frac{a^n}{a^n}$. In other words, it means *that some power of a has been divided by the same power of a.*

As, for example, $3^0 = \frac{3^2}{3^2} = \frac{3^6}{3^6} = \frac{3^m}{3^m} = \frac{3^n}{3^n} = 1$.

68. The Negative Exponent. — By the law of exponents in division $\frac{a^0}{a^n} = a^{0-n} = a^{-n}$.

But also $\frac{a^0}{a^n} = \frac{1}{a^n}$, since a^0 has already been proved equal to 1.

Therefore $a^{-n} = \frac{1}{a^n}$ since both are equal to the same thing.

Also $\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = a^n$.

Hence: *any quantity with a negative exponent is equal to the reciprocal of that quantity with the corresponding positive exponent.*

From the above principle it also follows that

Any **factor** *of the numerator of a fraction may be transferred to the denominator, or any* **factor** *of the denominator transferred to the numerator if the sign of its exponent is changed.*

Thus $\frac{a^2b^{-3}}{x^{-2}y} = \frac{a^2x^2}{b^3y}$.

Evidently it will always be possible to make the exponents of all **factors** positive by an application of the above rule.

Would it be correct to change $\frac{x^{-2}+y^2}{a^2+b^{-2}}$ to the form $\frac{b^2+y^2}{a^2+x^2}$? Why?

69. The Fractional Exponent. — We wish now to make clear the meaning of the power $a^{\frac{1}{r}}$, in which r is a positive integer, by assuming that it is in accordance with the law of exponents $a^m \cdot a^n = a^{m+n}$.

From this law $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Or by the law for involution $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2$ and we then obtain $(a^{\frac{1}{2}})^2 = a$.

Therefore $a^{\frac{1}{2}}$ is the number whose square is a, or $a^{\frac{1}{2}} = \sqrt{a}$. Stated in general terms

$$a^{\frac{1}{r}} \cdot a^{\frac{1}{r}} \cdot a^{\frac{1}{r}} \cdots r \text{ factors} = a^{\frac{1}{r}+\frac{1}{r}+\frac{1}{r}+\cdots r \text{ terms}} = a^{r \cdot \frac{1}{r}} = a.$$

Or by the laws for involution

$$a^{\frac{1}{r}} \cdot a^{\frac{1}{r}} \cdot a^{\frac{1}{r}} \cdots r \text{ factors} = (a^{\frac{1}{r}})^r = a.$$

Therefore $a^{\frac{1}{r}}$ is a number whose rth power is a.

Hence $$a^{\frac{1}{r}} = \sqrt[r]{a}.$$

Thus $$x^{\frac{1}{2}} = \sqrt{x}. \quad x^{\frac{1}{4}} = \sqrt[4]{x}. \quad 8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$$

Note that the denominator of the fractional exponent indicates the root and the numerator indicates the power of the radicand.

Applying the above laws we have

$$(4^{\frac{1}{2}})^2 = (\sqrt{4})^2 = 4. \qquad (-8^{\frac{1}{3}})^3 = (\sqrt[3]{-8})^3 = -8.$$

In general language $$(a^{\frac{1}{r}})^r = (\sqrt[r]{a})^r = a,$$

and $$(a^r)^{\frac{1}{r}} = \sqrt[r]{a^r} = a.$$

Therefore we can say $$(a^{\frac{1}{r}})^r = (a^r)^{\frac{1}{r}}$$

when we consider the positive root.

Again let us study the meaning of $a^{\frac{p}{r}}$ when $\frac{p}{r}$ is a positive or negative fraction. Assuming that p has the same sign as that of the fraction and that r is positive and the law

$$a^m \cdot a^n = a^{m+n}.$$

We have $$6^{\frac{2}{3}} \cdot 6^{\frac{2}{3}} \cdot 6^{\frac{2}{3}} = 6^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 6^2.$$

In other words, $6^{\frac{2}{3}}$ is that number whose cube is 6^2, or $6^{\frac{2}{3}} = \sqrt[3]{6^2}$. Or in general

$$a^{\frac{p}{r}} \cdot a^{\frac{p}{r}} \cdot a^{\frac{p}{r}} \cdots r \text{ factors} = a^{\frac{p}{r}+\frac{p}{r}+\frac{p}{r}+\cdots r \text{ terms}} = a^{r \cdot \frac{p}{r}} = a^p.$$

Therefore $a^{\frac{p}{r}}$ is the number whose rth power is a^p.

Hence $$a^{\frac{p}{r}} = \sqrt[r]{a^p}.$$

Examples:

$x^{\frac{3}{2}} = \sqrt{x^3}$. $x^{\frac{2}{3}} = \sqrt[3]{x^2}$. $12^{\frac{3}{4}} = \sqrt[4]{12^3}$. $8^{-\frac{2}{3}} = \sqrt[3]{8^{-2}} = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$.

The fractional exponent occurs as the result of applying the law of exponents in evolution whenever the index of the root is not contained evenly in the given exponent.

For example $\sqrt{x} = x^{\frac{1}{2}}$; $\sqrt{x^3} = x^{\frac{3}{2}}$; $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

From the above work you will notice that a fractional exponent implies that both a power and a root are to be taken, *the numerator indicating the power and the denominator the root.*

HISTORICAL NOTE

You may be interested to know that Nicole Oresme, a Norman bishop, grasped some idea of fractional exponents back in the fourteenth century. Also Chuquet, a Frenchman, about 1484, had some ideas concerning exponents and had written some expressions which used a form of the zero and negative exponents.

Simon Stevin, a Belgian (1548–1620), probably invented our present method of denoting powers. John Wallis, an Englishman (1616–1703), extended the use of fractional and negative exponents and showed some of their advantages.

It finally remained for Newton, however, to give us the present form of writing the different kinds of exponents.

Oral Exercises

Express each of the following with radical signs:

1. $x^{\frac{2}{3}}$.
2. $y^{\frac{1}{2}}$.
3. $m^{\frac{1}{2}}x^{\frac{3}{2}}$.
4. $a^{\frac{3}{4}}b^{\frac{5}{4}}$.
5. $x^{\frac{1}{3}}y^{\frac{3}{2}}$.
6. $a^{\frac{2}{3}}y^{\frac{1}{2}}z^{\frac{3}{5}}$.
7. $2 \cdot x^{\frac{1}{2}}$.
8. $3x^2 \cdot y^{\frac{1}{2}}$.

Express each of the following with fractional exponents:

9. $\sqrt{a^3}$.
10. $\sqrt[3]{m^5}$.
11. $\sqrt{3\,a^5}$.
12. $\sqrt[3]{2\,a^2y^4}$.
13. $\sqrt[4]{a^2x^3y^5}$.

Find the numerical value of each of the following:

14. $8^{\frac{1}{3}}$. **17.** $16^{\frac{3}{4}}$. **20.** $16^{\frac{5}{4}}$. **23.** $64^{\frac{2}{3}}$.

15. $16^{\frac{3}{4}}$. **18.** $27^{\frac{2}{3}}$. **21.** $25^{-\frac{1}{2}}$. **24.** $9^{\frac{3}{2}}$.

16. $4^{\frac{3}{2}}$. **19.** $9^{\frac{5}{2}}$. **22.** $25^{-\frac{3}{2}}$. **25.** $27^{\frac{4}{3}}$.

26. $\left(-\frac{8}{27}\right)^{\frac{2}{3}}$. **27.** $\left(\frac{.04}{.09}\right)^{\frac{1}{2}}$.

Write, using only positive exponents:

28. $2\,a^{-2}b^{-3}$. **29.** $x^3y^{-1}c^{-2}$.

30. $\frac{x^3}{3\,y^{-2}}$. **31.** $\frac{(2\,a)^{-3}b^2}{cd^{-2}}$. **32.** $\frac{x^2y^{-3}z^{-1}}{a^{-2}bc^{-5}}$.

Simplify: (The simplified form requires positive exponents.)

33. $\frac{a^0b^2c^3}{b^{-2}c^{-4}}$.

34. $(3\,a^2)^0 \cdot x^2y^{-2}$.

35. $\frac{a^{-3}b^2c}{ab^{-1}c^{-3}}$.

36. $\frac{x^ny^3z^{-4}}{3^{-1}x^{2n}}$.

37. $\frac{9^{\frac{1}{2}}a^{-2}}{b^{-3}}$.

38. $\frac{27^{\frac{2}{3}}ax^3}{2^{-1}a^{-1}x^{-3}}$.

39. $\frac{8^{-\frac{2}{3}}m^2n^{-2}y}{2^{-2}m^{-1}n^{-2}y^{-2}}$.

40. $\frac{5^{-2}(a-b)^0x^{-3}}{x^{-4}y^0}$.

41. $x^{\frac{3}{2}} \cdot x^{\frac{1}{2}}$.

42. $a^{\frac{1}{3}} \cdot b^{\frac{5}{3}}$.

43. $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}$.

44. $x^{\frac{2}{3}} \div x^{\frac{5}{3}}$.

45. $x^{\frac{1}{2}} \cdot x \cdot x^{\frac{3}{2}}$.

46. $a^{\frac{1}{3}} \cdot a^{\frac{1}{2}} \cdot a^{-\frac{1}{4}}$.

47. $\frac{a^{\frac{1}{5}} \cdot a^{-\frac{3}{10}}a^{-\frac{7}{15}}}{a^{\frac{1}{2}}}$.

48. $x^n \cdot x^{\frac{2}{3}}$.

49. $x^0 \cdot x^4 \cdot x^{-\frac{1}{3}}$.

50. $s^3 \div s^{-2}$.

51. $42\,t^{-3} \div t^4$.

52. $r^{-1} \div r^{-5}$.

53. $l^{-3} \div l$.

54. $3\,a^0(3\,a)^0$.

55. $x^{n-1} \cdot x^{1-n}$.

56. $\sqrt{x} \cdot \sqrt[3]{x}$.

57. $\sqrt{48a^0} \cdot \sqrt[3]{b^0}$.

58. $\left(\frac{a^{\frac{1}{3}}b^{\frac{1}{2}}}{a^{-\frac{2}{3}}}\right)^0$.

59. $(\sqrt{3\,a^2b^3})^0 \cdot 6(ab^2)^0$.

60. $a^{2n-3} \cdot a^{n+1}$.

61. $x^{-3+n} \div x^{2n-3}$.

62. $y^{4m+1} \div y^{4m-1}$.

63. $a^{-2} \div a^{n+2}$.

64. $x^2 \div \frac{1}{x^2}$.

65. $x^3 \div \frac{1}{x^{-3}}$.

66. $(x^{2a})^{-2} \div (x^{-2})^{2a}$.

67. $(x^{\frac{1}{2}})^3(x^3)^{-\frac{1}{2}}$.

68. $((a^2)^3)^{\frac{1}{6}}$.

69. $(x^{-2})^{n-2} \div x^4$.

70. $8^{\frac{2}{3}}x^6 \div 2\,x^{-2}$.

71. $a^3(a^2 + a - 3\,a^{-1})$.

72. $2\,a^{-2}(a^3 + a^2 + a)$.

73. $(e^x + e^{-x})e^{3x}$.

74. $(x^{\frac{2}{3}} + 3\,x^{\frac{1}{3}} + 4)(2\,x^{\frac{1}{3}})$.

75. $(a^2 - a^{-1}b^{-2})(a^{\frac{1}{2}}b^{\frac{1}{2}})$.

76. $(a - a^{-1})^2$.

77. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2$.

78. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

79. $(a^{\frac{1}{3}} + 2\,b^{-\frac{1}{3}})(a^{\frac{1}{3}} - 2\,b^{-\frac{1}{3}})$.

80. $(a^2b^{-1} - 1)^2$.

81. $(x^{-a} + y^{-b})(x^{-a} - y^{-b})$.

82. $(x^{-2} - 1)(x^{-2} + 3)$.

Exercises

Simplify each of the following:

1. $(16^{\frac{1}{2}})^{\frac{1}{2}}$.

2. $(-32)^{\frac{2}{5}}$.

3. $(64^{\frac{1}{3}})^{-2}$.

4. $(-27)^{\frac{4}{3}} \cdot 3^{-1}$.

5. $\frac{8^{\frac{2}{3}}}{8^0}$.

6. $(\frac{1}{4})^{-\frac{1}{2}} \cdot 8^{-\frac{1}{3}} \cdot 16^0$.

7. $\frac{3^2 \cdot 9^{-\frac{1}{2}}}{3^{-1}}$.

8. $9^{\frac{3}{2}} \cdot 4^{1.5} \div 36^{-\frac{1}{2}}$.

9. $0^3 \cdot 3^0 \cdot 9^{\frac{1}{2}}$.

10. $((\sqrt{16})^{\frac{3}{2}})^{-1}$.

11. $((\sqrt[3]{27})^2)^{\frac{1}{2}}$.

12. $\sqrt[4]{16^{-1}} \cdot 32^{-\frac{1}{5}}$.

13. $8^{\frac{2}{3}} + 25^{-\frac{1}{2}} - 64^{\frac{1}{6}} + \frac{1}{4^{-1}}$.

14. $\frac{(.16)^{\frac{1}{2}} \cdot .008^{\frac{1}{3}}}{(.25)^{-\frac{1}{2}}}$.

15. $(64\,a^{-6})^{-\frac{2}{3}}$.

16. $(-4\,a^2y^{-1}x^{-\frac{2}{3}})^{-1}$.

17. $(x^3\sqrt{x^{-1}})^{-2}$.

18. $\sqrt[5]{(x^{-1})^{-10} \cdot x^{-3}}$.

19. $\left(\frac{36\sqrt{xy^3}}{25\sqrt{x^3y}}\right)^{\frac{1}{2}}$.

20. $(a^n b^{-2n} c^{n^2})^{-n}$.

21. $\sqrt{x^{-2}z^{-1}} \cdot \sqrt[3]{x^{-1}\sqrt{z^3}}$.

22. $\sqrt[3]{8\,x^2\sqrt{y}} \div \sqrt{9\,y^3\sqrt[3]{x}}$.

23. $\sqrt[3]{a^{3b}x^{6c}} \cdot (a^c x^b)^0$.

24. $\sqrt{\left(\left(\frac{16}{49}\right)^{-\frac{1}{2}}\right)^3}$.

25. $\left(\frac{a^{n+1}}{a^{1-n}}\right)^n \div \frac{a^{-n}}{a}$.

26. $\sqrt[3]{27^{\frac{1}{2}} \cdot 2^{-3}} \cdot \sqrt{9^3}$.

27. $\left[\frac{a^{\frac{1}{2}}x^{-\frac{1}{2}}}{(ax)^{\frac{2}{3}}} \div \frac{a^{-\frac{1}{2}}x}{x^{\frac{2}{3}}a^0}\right]^{-1}$.

28. $\left(\frac{a^3b^{-4}}{x^{-1}y^2}\right)^2 \cdot \frac{x^{-1}b^{-1}}{a^{\frac{3}{4}}y^{\frac{3}{2}}}$.

29. $\sqrt[4]{x^{2n+1}} \cdot \sqrt[4]{x^{-1}}$.

30. $\sqrt[3]{\sqrt{(16^{-6})^2}}$.

31. $\frac{\sqrt[4]{x^{-2}\sqrt{y}}}{\sqrt{x^{-4}y^{\frac{1}{8}}}}$.

32. $\left[\sqrt[4]{\frac{y^{\frac{1}{2}}\sqrt{xy}}{x^{\frac{2}{3}}}}\right]^{-\frac{2}{3}}$.

33. $\frac{2}{2^{-1}+2^{-2}}$.

34. $\frac{3^{-1}}{3^{-1}+3^{-2}}$.

35. $\frac{2^{-1}+3^{-1}}{2^{-1}-3^{-1}}$.

36. $\frac{a^2+b^{-2}}{a^{-2}+b^2}$.

37. $\frac{a^3+b^{-3}}{a+b^{-1}}$.

38. $\frac{a^{-2}x^{-2}}{(ax)^{-1}-(ax)^{-2}}$.

39. $\frac{a^{-2}-3^{-2}}{a^{-1}+3^{-1}}$.

40. $\frac{x^{-4}-y^{-4}}{x^{-2}+y^{-2}}$.

41. $\frac{x^{-6}-64\,y^{-6}}{x^{-3}-8\,y^{-3}}$.

42. $10^{-6} \cdot 64000000$.

Hint: Note that you have the same result as you would obtain if you were to move the decimal point as many places to the left as there are units in the negative exponent.

43. $10^{-4} \cdot 48000$.

44. $10^{-5} \cdot 500000$.

45. $10^{-2} \cdot 100$.

46. $10^{-3} \cdot 10^3$.

47. $10^{-4} \cdot 10^6$.

What is the value of

48. $27^{\frac{2}{3}} - 16^{\frac{3}{4}} + 64^0$?

49. $81^{-\frac{1}{2}} + (-32)^{\frac{4}{5}} + 9^{\frac{3}{2}} + 1564^0$?

50. $\dfrac{7 \cdot 13^0 \div 7}{21^0} + 3^0 \cdot \dfrac{4^0 \cdot 7^0}{(7a+b)^0} + 8^{-\frac{2}{3}}$.

51. Express each of the following as a power of 2:

$$8^3,\ 4^5,\ 4^3,\ 8^{\frac{2}{3}},\ 16^{\frac{3}{4}}.$$

52. $(.1)^{-1} + (.125)^{\frac{1}{3}} + 8 \cdot 27^{\frac{2}{3}} = ?$

53. Does $16^0 + 16^{-1} + 16^{\frac{1}{2}} - 16^{-\frac{3}{4}}$ equal or not equal $1\frac{3}{16}$? Prove.

Simplify:

54. $x^{3.2} + x^{-3.2}$.

55. $\dfrac{(x^a)^{a+3}}{x^{a+2} \cdot x^{a-4}}$.

56. $(\sqrt[n]{x^{-a}})^{-3}$.

57. $\dfrac{x^{-n}y^{-n}}{\frac{1}{2}z} \div \dfrac{z^{-1}}{x^{-2n}y^{-2n}}$.

58. $(-3^{-2}x^{-3}y^{-6})^{-\frac{1}{3}}(-3^{-\frac{1}{3}}x^{-\frac{1}{3}}y^{-1})^2$.

70. Arrangements of Terms. — When we perform the operations of multiplication, division, or square root upon polynomials having negative or fractional exponents, arrangement of the terms according to powers of a common letter is valuable. In arranging, you should notice that the zero power of a letter does not contain the letter since it equals 1. Also negative powers are lower than the zero power. Notice also that -3 is less than -2, etc.

Illustrative example I.

Arrange in descending powers of x

$$x^3 + \frac{1}{x} + \frac{5}{x^2} + x - 6x^{-3} + 4x^2 - 2.$$

The arrangement is

$$x^3 + 4x^2 + x - 2 + x^{-1} + 5x^{-2} - 6x^{-3}.$$

Or, if we prefer to use only positive exponents,

$$x^3 + 4x^2 + x - 2 + \frac{1}{x} + \frac{5}{x^2} - \frac{6}{x^3}.$$

Illustrative example II.

Multiply $a^{-1} + a^{-\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{-1} - a^{-\frac{1}{2}}b^{\frac{1}{2}} + b$.

Solution:

$$\begin{array}{l} a^{-1} + a^{-\frac{1}{2}}b^{\frac{1}{2}} + b \\ \underline{a^{-1} - a^{-\frac{1}{2}}b^{\frac{1}{2}} + b} \\ a^{-2} + a^{-\frac{3}{2}}b^{\frac{1}{2}} + a^{-1}b \\ \quad - a^{-\frac{3}{2}}b^{\frac{1}{2}} - a^{-1}b - a^{-\frac{1}{2}}b^{\frac{3}{2}} \\ \underline{\qquad\qquad + a^{-1}b + a^{-\frac{1}{2}}b^{\frac{3}{2}} + b^2} \\ a^{-2} \qquad + a^{-1}b \qquad + b^2 \end{array}$$

Illustrative example III.

Divide $16\,x - 8\sqrt{y} - \dfrac{2\sqrt{y}}{x^{\frac{3}{2}}} + \dfrac{y}{x^{\frac{5}{2}}}$ by $\dfrac{1}{\sqrt{xy}} - \dfrac{8\,x^2}{y}$.

Solution: Using exponents to express radicals and fractions we have

$$\begin{array}{l|l} 16\,x - 8\,y^{\frac{1}{2}} - 2\,x^{-\frac{3}{2}}y^{\frac{1}{2}} + x^{-\frac{5}{2}}y & -8\,x^2y^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} \\ \underline{16\,x \qquad - 2\,x^{-\frac{3}{2}}y^{\frac{1}{2}}} & \underline{-2\,x^{-1}y + x^{-2}y^{\frac{3}{2}}}, \text{ quotient.} \\ \quad -8\,y^{\frac{1}{2}} \qquad + x^{-\frac{5}{2}}y & \\ \quad \underline{-8\,y^{\frac{1}{2}} \qquad + x^{-\frac{5}{2}}y} & \end{array}$$

Illustrative example IV.

Extract the square root of

$$a^{-1} + 4\,a^{-\frac{1}{2}}b + 4\,b^2 - 2\,b^{\frac{1}{2}} - 4\,a^{\frac{1}{2}}b^{\frac{3}{2}} + ab.$$

Solution:

$$\begin{array}{l|l} a^{-1}+4\,a^{-\frac{1}{2}}b+4\,b^2-2\,b^{\frac{1}{2}}-4\,a^{\frac{1}{2}}b^{\frac{3}{2}}+ab & a^{-\frac{1}{2}}+2\,b-a^{\frac{1}{2}}b^{\frac{1}{2}} \\ \underline{a^{-1}} & \\ 2\,a^{-\frac{1}{2}}+2\,b & +4\,a^{-\frac{1}{2}}b+4\,b^2 \\ \quad \underline{+4\,a^{-\frac{1}{2}}b+4\,b^2} & \\ \quad 2\,a^{-\frac{1}{2}}+4\,b-a^{\frac{1}{2}}b^{\frac{1}{2}} & -2\,b^{\frac{1}{2}}-4\,a^{\frac{1}{2}}b^{\frac{3}{2}}+ab \\ \quad \underline{\qquad\qquad -2\,b^{\frac{1}{2}}-4\,a^{\frac{1}{2}}b^{\frac{3}{2}}+ab} & \end{array}$$

Exercises

1. Arrange $x^{\frac{2}{3}} + 2x^{5} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$ in descending order of the powers of x.

2. Arrange $x^{-7} + x^{-2} - x^{-3} + x^{-4} + x$ in ascending order of exponents.

Multiply:

3. $(x^{\frac{1}{3}} + a^{-2})(x^{\frac{1}{3}} - a^{-2})$.
4. $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$.
5. $(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{1}{3}} + y^{\frac{1}{3}})$.
6. $2x^{\frac{1}{2}} - 3 + x^{-\frac{1}{2}}$ by $4x^{\frac{1}{2}} + 5$.
7. $3x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 6 + x^{-\frac{1}{3}}$ by $2x^{\frac{1}{3}} - 3$.
8. $4a^{\frac{3}{4}} - 2a^{\frac{1}{4}} + 6a^{-\frac{1}{4}} - 5a^{-\frac{3}{4}}$ by $a^{\frac{1}{4}} - 2 + 3a^{-\frac{1}{4}}$.
9. $e^{x} + 3 + e^{-x}$ by $2e^{x} - 3e^{-x}$.
10. $m^{\frac{1}{2}} + m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}}$ by $m^{\frac{1}{2}} - m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}}$.
11. $4a^{-3} + 6a^{-2} + a^{-1} + 3$ by $a^{-1} + 5$.
12. $2x^{-n} + 3x^{n} - 4x^{3n}$ by $5x^{-2n} - 3x^{-n}$.
13. $3x^{\frac{1}{2}} - 5x^{\frac{1}{4}}y - 6y^{2} + 2x^{-\frac{1}{4}}y^{3}$ by $x^{\frac{1}{4}} + 2y$.

Divide:

14. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.
15. $x^{\frac{1}{3}} - y^{\frac{1}{3}}$ by $x^{\frac{1}{6}} - y^{\frac{1}{6}}$.
16. $x^{-1} + y^{-1}$ by $x^{-\frac{1}{3}} + y^{-\frac{1}{3}}$.
17. $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{4}} - a^{\frac{1}{8}}b^{\frac{1}{8}} + b^{\frac{1}{4}}$.
18. $5x + 2x^{\frac{3}{3}} - 2x^{\frac{1}{3}} + 1$ by $5x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 1$.
19. $6x^{2} - 7x^{\frac{5}{3}} - 19x^{\frac{4}{3}} + 5x + 9x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$ by $3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}$.
20. $3x^{-10} + x^{6} - 4x^{-6}$ by $2x^{-2} + x^{2} + 3x^{-6}$.
21. $a^{-\frac{1}{2}} - a^{-\frac{1}{6}} - 2a^{\frac{1}{6}} + 5$ by $1 + 2a^{\frac{1}{6}}$.

22. $1 - a^{\frac{1}{2}} + 2\,a^2 - 2\,a$ by $1 - \sqrt{a}$.

23. $\dfrac{1}{\sqrt[3]{x^4}} + \dfrac{1}{y^2\sqrt[3]{x^2}} + \dfrac{1}{y^4}$ by $\dfrac{1}{x} + \dfrac{1}{y\sqrt[3]{x^2}} + \dfrac{1}{y^2\sqrt[3]{x}}$.

Extract the square root of:

24. $x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 2$.

25. $x + 6\,x^{\frac{1}{2}} + 17 + 24\,x^{-\frac{1}{2}} + 16\,x^{-1}$.

26. $4\,a^3 + 36\,a^2 + 25\,a - 16\,a^{\frac{5}{2}} - 40\,a^{\frac{3}{2}}$.

27. $x - 2\,x^{\frac{3}{4}}y^{\frac{1}{4}} + y + 3\,x^{\frac{1}{2}}y^{\frac{1}{2}} - 2\,x^{\frac{1}{4}}y^{\frac{3}{4}}$.

28. $x^{-1} + y^{\frac{2}{3}} + 2\,x^{-\frac{1}{2}}y^{\frac{1}{3}} + xy^{\frac{4}{3}} - 2\,y^{\frac{2}{3}} - 2\,x^{\frac{1}{2}}y$.

29. $4\,a^x + 28 - 24\,a^{-\frac{x}{2}} + 9\,a^{-x} - 16\,a^{\frac{x}{2}}$.

30. $.01\,x^2 + .1\,xy^{-1} + \frac{1}{4}\,y^{-2}$.

Review

1. Find the side of a square whose diagonal is 10 inches, correct to the nearest tenth.

2. What is the shortest distance from one corner to the opposite corner of a rectangular lot 80 rods by 30 rods?

3. Which is greater $(2^{-4})^5$ or $2^{-4} \cdot 2^5$? How much greater?

4. $\dfrac{(a^2 - b^2) + (a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{3}{2}})}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} = ?$

Divide numerator and denominator by the common factor.

5. Change to positive exponents and simplify:

$$\frac{x^{-1} + 2\,y^{-1}}{x^{-3} + 8\,y^{-3}}.$$

6. The sales of some chain stores in the United States for two successive years are given in round numbers of millions. Draw bar graphs, October in black and November in green, totals in black dashed line, to illustrate these facts.

	Stores #1		Stores #2		Stores #3		Stores #4		Stores #5	
	1923	1922	1923	1922	1923	1922	1923	1922	1923	1922
Nov.	17.2	14.8	20.4	20.1	14.1	11.	7.5	6.3	7.1	5.7
Oct.	18.1	15.7	22.5	19.9	15.1	10.2	7.2	6.	7.5	5.9
11 Mo.	160.	140.	194.	161.	119.	81.	68.	55.	53.	43.

If the profits were computed at 10% of the sales, represent the profits in each case with a red line.

7. Soundings were taken in a river at intervals of 20 feet and found to be

9.5 ft. 18 ft. 20 ft. 22 ft. 14 ft. 12 ft. 20 ft. 22 ft. 15 ft. 6 ft.

Construct a curve chart to represent these depths. If a boat draws 15 feet of water, show where it must go to navigate the stream safely at this point.

71. Fractional and Negative Exponents in Equations. — Occasionally it becomes necessary to solve equations containing fractional or negative exponents, such as $x^{\frac{2}{3}} = 4$ or $x^{-\frac{1}{3}} = -2$.

Such equations are solved by the following

Rule. — (1) *Simplify the equation so that the left member contains only the unknown quantity with coefficient one.*

(2) *Raise both members of the equation to such a power that the exponent of the unknown shall become one.*

(3) *Simplify the right member of the equation.*

Illustrative example I.

Solve $x^{\frac{2}{3}} = 4.$

Solution:

Raise both members of the equation to the $\frac{3}{2}$ (three halves) power.

Then $(x^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}$.

$\therefore x = (\pm 2)^3$

$= \pm 8$.

Check in original equation.

$(\pm 8)^{\frac{2}{3}} = 4$.

$(\pm 2)^2 = 4$.

$4 = 4$.

Illustrative example II.

Solve $x^{-\frac{1}{3}} = 2$.

Solution:

Raise both members of the equation to the -3 (minus 3) power.

Then $(x^{-\frac{1}{3}})^{-3} = 2^{-3}$.

$\therefore x = \frac{1}{8}$.

Illustrative example III.

Solve $\frac{1}{2}x^{-2} = 2$.

Solution:

Divide both members of the equation by $\frac{1}{2}$ or multiply both members of the equation by 2 and we obtain

$x^{-2} = 4$.

Hence $(x^{-2})^{-\frac{1}{2}} = 4^{-\frac{1}{2}}$,

and $x = \pm \frac{1}{2}$.

Exercises

Solve each of the following equations:

1. $x^{\frac{1}{2}} = 3$.
2. $x^{\frac{1}{3}} = 3$.
3. $x^{\frac{3}{2}} = 8$.
4. $x^{\frac{1}{2}} = 2$.
5. $x^{\frac{1}{4}} = 2$.
6. $x^{\frac{1}{3}} = 2$.
7. $x^{\frac{1}{5}} = 2$.
8. $x^{\frac{1}{3}} = \frac{1}{2}$.
9. $x^{\frac{1}{2}} = \frac{1}{4}$.
10. $x^{\frac{1}{3}} = \frac{2}{3}$.
11. $x^{\frac{1}{2}} = \frac{1}{5}$.
12. $x^{\frac{1}{2}} = \frac{1}{7}$.
13. $\sqrt{x} = 3$.
14. $\sqrt{x} = b$.
15. $x^{\frac{1}{2}} = h$.
16. $x^{\frac{1}{3}} = k$.

17. $x^{\frac{1}{4}} = m.$
18. $\sqrt[3]{x} = a.$
19. $\sqrt[5]{x} = h.$
20. $x^{\frac{1}{2}} = 3\ h.$
21. $x^{\frac{1}{2}} = 4\ k.$
22. $\sqrt[3]{x} = 5\ h.$
23. $x^{\frac{1}{3}} = 3\ h.$
24. $x^{\frac{1}{4}} = 2\ a.$
25. $x^{\frac{1}{2}} = \dfrac{1}{h}.$
26. $x^{\frac{1}{6}} = \dfrac{1}{b}.$
27. $\sqrt[3]{x} = \dfrac{2}{b}.$
28. $x^{\frac{1}{3}} = \dfrac{2\ a}{b}.$
29. $x^{\frac{1}{4}} = \dfrac{2}{3\ b}.$
30. $x^{-\frac{1}{3}} = \frac{2}{3}.$
31. $x^{\frac{2}{3}} = 9.$
32. $x^{\frac{3}{2}} = -64.$
33. $x^{-\frac{1}{5}} = -2.$
34. $x^{-\frac{2}{3}} = 16.$
35. $x^{-\frac{3}{4}} = \frac{1}{8}.$
36. $x^{\frac{3}{5}} = -\frac{1}{8}.$
37. $\dfrac{1}{x^{\frac{1}{3}}} = \dfrac{2}{3}.$
38. $\dfrac{1}{x^{\frac{1}{2}}} = \dfrac{x^2}{32}.$
39. $\dfrac{x^{\frac{1}{3}}}{9} = \dfrac{9}{x}.$
40. $x^{-\frac{1}{n}} = a.$
41. $x^{\frac{1}{2}} + 2 = 4.$
42. $x^{\frac{1}{2}} + 3 = 6.$
43. $x^{\frac{1}{3}} - 2 = 2.$
44. $x^{\frac{1}{4}} + 2 = 4.$
45. $x^{\frac{1}{5}} + 5 = 7.$
46. $x^{\frac{1}{6}} + 9 = 11.$
47. $x^{\frac{1}{3}} + 3 = 6.$
48. $x^{\frac{2}{3}} + 3 = 7.$

Hint to Example 48: First reduce the equation so that all terms without the unknown are combined on the right side.

49. $x^{\frac{3}{4}} + 7 = 15.$
50. $x^{\frac{5}{6}} + 5 = 37.$
51. $2^x = 16.$

Hint: $2^x = 2^4.$
What does $x =$?

52. $2^x = 64.$
53. $\dfrac{1}{2^{-x}} = 32.$
54. $\dfrac{1}{3^{-x}} = 27.$
55. $4^x = 64.$
56. $3^{-x} = \frac{1}{9}.$
57. $10^{-x} = \frac{1}{10}.$
58. $10^{-x} = \frac{1}{100}.$
59. $10^{-x} = \frac{1}{1000}.$
60. $(x-2)^{\frac{1}{2}} = 3.$
61. $(x-3)^{\frac{1}{3}} = 2.$
62. $(x^2 - 4\ x + 4)^{\frac{1}{2}} = 1.$
63. $(x^2 + 2\ x + 5)^{\frac{1}{2}} = x + 3.$
64. $x^{\frac{1}{2}} - 4 = b.$
65. $x^{\frac{1}{2}} - b = 4.$
66. $(x+b)^{\frac{1}{2}} = (3\ b)^{\frac{1}{2}}.$
67. $(x-a)^{\frac{1}{2}} = (4\ a)^{\frac{1}{2}}.$

CHAPTER VII

RADICALS

In your work in geometry and other sciences you will frequently find it necessary to deal with quantities which are irrational, that is, indicated roots which cannot be extracted exactly.

In the last chapter you observed that such roots may be indicated by the fractional exponent. They are, however, more frequently indicated by the radical sign, the use of which will be explained in the present chapter.

72. Irrational Quantities. — A *rational number* is a positive or negative integer or the indicated quotient of such integers. Other numbers are *irrational.*

Thus, $3, \ -7, \ \frac{2}{3}, \ .3, \ -\frac{8}{5}$ are rational numbers.

$\sqrt{3}, \ \sqrt[3]{5}, \ \sqrt{1+\sqrt{2}}$ are irrational numbers.

An indicated root which cannot be exactly extracted is always *irrational.*

The *radicand* is the number under the radical sign.

A *surd* is the simplest type of an irrational number, that in which the radicand is rational. Thus, $\sqrt{5}, \ \sqrt[3]{9}, \ \sqrt[4]{6}, \ \sqrt{\frac{1}{2}}$ are surds, while the more complicated expression $\sqrt[3]{3-2\sqrt{5}}$ is not a surd.

The *order* of a radical is indicated by its index. Thus, $\sqrt{5}, \ 2\sqrt{11}, \ \sqrt{15}, \ \sqrt{x+4}$ are radicals of the second order, while $\sqrt[3]{6}, \ 5\sqrt[3]{19}$ are of the third order. The order is also the denominator of the fractional exponent.

A *quadratic surd* is a surd of the second order, *i.e.* $\sqrt{5}$, $\sqrt{xy}$.

An *entire surd* is a surd which has unity as its coefficient, as $\sqrt{3}$, $\sqrt[3]{6}$, $(x+4)^{\frac{1}{3}}$.

A *mixed surd* is a surd which has a rational coefficient, as $5\sqrt{3}$, $4\sqrt[3]{24}$, $3(x+4)^{\frac{1}{3}}$.

The *coefficient* of a radical is the number prefixed to the radical to show how many times it is taken; thus 5 is the coefficient of $5\sqrt{3}$, and $4(x+y)$ of $4(x+y)\sqrt{a}$.

Oral Exercise

1. State which of the following are rational, and which are irrational:

2, $6\frac{3}{4}$, $\sqrt{7}$, 11, $\frac{3}{4}$, $\sqrt[3]{6}$, $\sqrt[3]{5-\sqrt{2}}$, $\sqrt{1}$, $\sqrt[3]{\frac{1}{8}}$, 17, $5^{\frac{1}{2}}$, $16^{\frac{1}{4}}$, $(x+y)^{\frac{1}{2}}$.

2. State the radicand in each of the following radicals:

$3\sqrt{11}$, $2\sqrt[3]{26}$, $\sqrt[3]{104}$, $\sqrt[5]{3-\sqrt{6}}$, $6-\sqrt{5}$, $3\sqrt[4]{17}$, $2\sqrt{5-3\sqrt[3]{2}}$, $\sqrt{171}$.

3. State the order of the above radicals.

4. State which of the above radicals are also surds.

5. Which of the following are quadratic surds?

$\sqrt{5}$, $\sqrt[3]{4}$, $(x+y)^{\frac{1}{2}}$, $(x+y)^{\frac{1}{3}}$, $\sqrt{16}$, $\sqrt[4]{41}$, $\sqrt{ab}$.

6. State which of the following are entire surds:

$\sqrt{5}$, $\sqrt{4}$, $(x+y)^{\frac{1}{3}}$, $(x+y)^{\frac{1}{2}}$, $\sqrt{16}$, $6\sqrt{5}$, $7\sqrt{7}$.

7. State which of the following are mixed surds:

$8\sqrt{5}$, $\sqrt{25}$, $3(x+y)^{\frac{1}{2}}$, $5\sqrt{7}$, $4\sqrt{16}$, $9\sqrt{8}$, 7.

8. State the coefficient in each of the following surds:

$\sqrt{5}$, $4\sqrt{6}$, $7(x+y)^{\frac{1}{2}}$, $-2\sqrt{3}$, $.5\sqrt{7}$, $\frac{1}{2}\sqrt{ab}$.

73. Simplification of Radicals. — We shall now state some principles of radicals which the pupil should memorize.

1. If the root indicated by the radical sign can be extracted, this should be done, thus getting rid of the radical sign entirely.

2. The order of the radical should be made as low as possible.

3. The radicand should be made as small as possible.

4. Fractional radicands should not be permitted.

NOTE. There are exceptions in trigonometry.

5. A radical should not be left in the denominator of a fraction.

NOTE. There may be some exceptions in trigonometry.

74. First Principle of Simplification. — As an illustration of the first principle we note that $\sqrt{\frac{4}{9}}$, $\sqrt{144}$, and $\sqrt[3]{\frac{1}{8}}$ should be written $\frac{2}{3}$, 12, and $\frac{1}{2}$, respectively.

75. Second Principle of Simplification. — We shall now explain how the second principle is applied. You should first note that a fourth root is obtained by extracting the square root, and then extracting the square root of that result. Similarly a sixth root may be obtained by extracting the square root of the cube root, or the cube root of the square root.

$\sqrt[4]{4} = \sqrt{2}$, for we may extract the square root of four once, obtaining 2, and the second extraction we must then indicate as $\sqrt{2}$, or by the use of fractional exponents.

$$\sqrt[4]{4} = 4^{\frac{1}{4}} = (2^2)^{\frac{1}{4}} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$$

Similarly, $\sqrt[6]{27} = \sqrt{3}$, for we may extract the cube root of 27, obtaining 3, while the square root remains to be obtained. Or, by fractional exponents,

$$\sqrt[6]{27} = 27^{\frac{1}{6}} = (3^3)^{\frac{1}{6}} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}.$$

Hence in general $\sqrt[q]{a^{nq}} = a^{nq/q} = a^n$.

Exercises

Simplify each of the following expressions:

1. $\sqrt[3]{8}$.
2. $\sqrt[4]{\frac{1}{16}}$.
3. $\sqrt[4]{81}$.
4. $\sqrt{125}$.
5. $\sqrt[4]{625}$.
6. $\sqrt[4]{\frac{1}{64}}$.
7. $\sqrt[3]{216}$.
8. $\sqrt[3]{.008}$.
9. $\sqrt[4]{.0016}$.
10. $\sqrt[5]{32\,x^5}$.
11. $\sqrt[6]{64\,x^{12}y^6}$.
12. $\sqrt[n]{x^{3n}}$.
13. $\sqrt[4]{9}$.
14. $\sqrt[6]{8}$.
15. $\sqrt[4]{25}$.
16. $\sqrt[4]{16}$.
17. $\sqrt[6]{125}$.
18. $\sqrt[6]{64}$.
19. $\sqrt[4]{36}$.
20. $\sqrt{a^{2n}}$.
21. $\sqrt[3]{128}$.
22. $\sqrt[3]{-8}$.
23. $\sqrt[5]{-32}$.
24. $\sqrt[3]{-125}$.
25. $\sqrt[6]{4}$.
26. $\sqrt[6]{9}$.
27. $\sqrt[6]{16}$.
28. $\sqrt[6]{25}$.
29. $\sqrt{361}$.
30. $\sqrt[4]{49}$.
31. $\sqrt[6]{49}$.
32. $\sqrt[4]{100}$.
33. $\sqrt[6]{144}$.
34. $\sqrt[6]{100}$.
35. $\sqrt[4]{121}$.
36. $\sqrt[6]{121}$.
37. $\sqrt[4]{16\,x^4}$.
38. $\sqrt[4]{169}$.
39. $\sqrt[6]{169}$.
40. $\sqrt[4]{144}$.
41. $\sqrt[4]{a^{4n}}$.
42. $\sqrt[6]{a^{12n}}$.
43. $\sqrt[3]{-3\frac{3}{8}}$.

44. How would you extract an 8th root? A 9th root? Can you formulate a rule for this type of simplification?

76. The Third Principle of Simplification may be shown by the following:

Illustrative examples:

I. $\sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$.

II. $\sqrt{300} = \sqrt{100 \cdot 3} = 10\sqrt{3}$.

III. $\sqrt[3]{56} = \sqrt[3]{8 \cdot 7} = 2\sqrt[3]{7}$.

IV. $\frac{1}{4}\sqrt{128} = \frac{1}{4}\sqrt{64 \cdot 2} = \frac{1}{4} \cdot 8\sqrt{2} = 2\sqrt{2}$.

Rule. — (1) *Factor the radicand into two factors, one of which is the largest possible power whose degree is equal to the order of the radical under consideration.*

(2) *Extract the indicated root of that factor and write it as a coefficient of the radical whose radicand is the other factor.*

Illustrative example:

Simplify $\sqrt[4]{48}$.

Solution: Since this radical is of the *fourth* order we must find a factor of 48 which is an exact *fourth* power.

This factor is 16, the fourth power of 2.

We now have $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = 2\sqrt[4]{3}$.

77. The Fourth Principle of Simplification which treats of fractional radicands is applied as in the following:

Illustrative examples:

I. Simplify $\sqrt{\frac{3}{8}}$.

Solution: $\sqrt{\frac{3}{8}} = \sqrt{\frac{3}{8} \cdot \frac{2}{2}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16} \cdot 6} = \frac{1}{4}\sqrt{6}$.

Here we have multiplied both numerator and denominator of the given fraction by 2. This does not change the value of the given fraction, but makes the denominator a perfect square. The fraction $\frac{6}{16}$ is then factored in such a way that one factor shall be a perfect square, and the radical simplified according to the rule of the preceding paragraph.

II. $\sqrt{1\frac{1}{3}} = \sqrt{\frac{4}{3} \cdot \frac{3}{3}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{9} \cdot 3} = \frac{2}{3}\sqrt{3}$.

III. $\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{3}{4} \cdot \frac{2}{2}} = \sqrt[3]{\frac{6}{8}} = \sqrt[3]{\frac{1}{8} \cdot 6} = \frac{1}{2}\sqrt{6}$.

Why did we multiply the fraction in the second step of III by $\frac{2}{2}$?

Note that in the fourth step of III, we factor so that one of our factors is a perfect cube.

Rule. — (1) *If the radicand is a mixed number, reduce it to a common fraction.*

(2) *Multiply both numerator and denominator of this fraction by the smallest number which will make the denominator a perfect power of a degree equal to the order of the radical.*

(3) *Factor and simplify according to the rule of the preceding section.*

Exercises

Simplify each of the following expressions:

1. $\sqrt{32}$.
2. $\sqrt{75}$.
3. $\sqrt{72}$.
4. $\sqrt{162}$.
5. $\sqrt{80}$.
6. $\sqrt{8\,x^2}$.
7. $\sqrt{50\,x^3}$.
8. $\sqrt[3]{81}$.
9. $\sqrt[3]{16}$.
10. $\sqrt[3]{54\,x^3}$.
11. $\sqrt[3]{32\,x^6}$.
12. $\sqrt[3]{40\,x^3y^3}$.
13. $\sqrt[4]{32}$.
14. $\sqrt[4]{162}$.
15. $\sqrt[4]{625\,x^4y}$.
16. $-\sqrt[3]{192}$.
17. $2\sqrt{18}$.
18. $3\sqrt{40\,x^2}$.
19. $\frac{1}{2}\sqrt{16\,x}$.
20. $\frac{3}{4}\sqrt{32\,x^2y^2z}$.
21. $-2\sqrt{32}$.
22. $-\frac{1}{3}\sqrt{27\,x^2yz^2}$.
23. $4\sqrt[3]{16}$.
24. $4\sqrt[4]{16}$.
25. $\frac{1}{4}\sqrt[5]{64}$.
26. $\sqrt[3]{-54\,x^3}$.
27. $\sqrt[3]{-40\,x^6}$.
28. $ab\sqrt{9\,b^3}$.
29. $ab\sqrt{a^2b^2c}$.
30. $-5\sqrt{8\,a^2}$.
31. $\sqrt{\sqrt{9}}$.
32. $\sqrt[m]{x^{2m}}$.
33. $\sqrt[7]{128\,a^7b^9}$.
34. $\frac{1}{2}\sqrt{32}$.
35. $\frac{1}{3}\sqrt{45\,a^2}$.
36. $-\frac{1}{4}\sqrt[3]{-16}$.
37. $\frac{1}{4}\sqrt[3]{48\,x^3}$.
38. $\sqrt{(x^2+2\,x+1)y}$.
39. $\sqrt{(x^2-4\,y^2)(x+2\,y)}$.
40. $\sqrt{36\,x^2(a+1)^3}$.
41. $4\sqrt{\frac{1}{8}}$.
42. $3\sqrt[3]{\frac{1}{3}}$.
43. $\frac{1}{5}\sqrt{\frac{2}{5}}$.
44. $-6\sqrt{\frac{7}{3}}$.

45. $\sqrt{\frac{x^2}{12}}$.

46. $\sqrt{\frac{3x}{5y}}$.

47. $\sqrt{4x - 4y}$.

48. $\sqrt[3]{8x + 8y}$.

49. $4a\sqrt{\frac{16x + 16y}{36}}$.

50. $3\sqrt{\frac{x^2y}{27}}$.

51. $\sqrt{1\frac{3}{5}}$.

52. $-9\sqrt{1\frac{2}{3}}$.

53. $\sqrt{1 - (\frac{1}{3})^2}$.

54. $\sqrt{2 - (\frac{1}{2})^2}$.

55. $\sqrt{3a^2 + \left(\frac{a}{2}\right)^2}$.

56. $\sqrt{a^2 + \frac{a^2}{4}}$.

57. $\sqrt{25 + 75\sqrt{3}}$.

58. $\sqrt{48 \cdot 32}$.

59. $\sqrt[3]{\frac{x}{27y}}$.

60. $\sqrt[3]{\frac{128x^7y^3}{a^6x^{13}}}$.

61. $\sqrt[4]{\frac{16x^5y^{12}}{z^3x^9}}$.

62. $\sqrt[n]{\frac{x^{3n}}{y^{n-1}}}$.

63. $\sqrt{\frac{x^{4n}}{y^{3n-2}}}$.

64. $\sqrt{.27}$.

65. $\sqrt[3]{.016}$.

66. If $\sqrt{2} = 1.414$, find the value of each of the following:
(a) $\sqrt{8}$. (b) $\sqrt{32}$. (c) $\sqrt{2\pi^2}$. (d) $\sqrt{8r^2}$ if $r = 10$.

Introduce the coefficients of the following radicals under the radical sign, thus forming entire surds.

67. $2\sqrt{3}$.

Hint: What number, used as a factor, under the square root sign is equivalent to the coefficient 2?

68. $5\sqrt{3}$.

69. $2\sqrt[3]{4}$.

70. $3\sqrt[3]{5}$.

71. $3a\sqrt{5a}$.

72. $5a^2\sqrt[3]{3a}$.

73. $2a^2\sqrt[4]{4a}$.

74. $\frac{1}{3}\sqrt[3]{\frac{9x^2}{4}}$.

75. $\frac{x}{y}\sqrt[n]{\frac{x^{1+2n}}{y}}$.

76. $\frac{1}{3}\sqrt[3]{54\,x^2}$.

77. $\frac{x}{3}\sqrt[3]{\frac{9}{x^2}}$.

78. $(1+x)\sqrt{\frac{1-x}{1+x}}$.

79. $(a-1)\sqrt{\frac{2}{a-1}+1}$.

80. $(2\,x+1)\sqrt{\frac{2}{4\,x^2-1}}$.

81. $\frac{a^2-1}{a^2+1}\sqrt{1-\frac{2\,a}{(a+1)^2}}$.

First simplify the following radicals, then evaluate them with the following values: $x = 1$, $y = 2$, $z = 3$, $w = 0$. If any radicals remain, find their value to the nearest hundredth.

82. $\sqrt{x^2y^2}$.

83. $\sqrt{x^2z^2}$.

84. $\sqrt[3]{z^3w}$.

85. $\sqrt{y^3z^2}$.

86. $5\sqrt{x^2y^2z^3}$.

87. $\frac{1}{3}\sqrt[3]{z^3y^6}$.

88. $.5\sqrt{x^4y^4z}$.

89. $\sqrt{\frac{x+y}{z^2}}$.

90. $\sqrt{\frac{xyw}{x+z}}$.

78. Addition and Subtraction of Radicals. — Before taking up the fifth principle of simplification which is connected with division, we must learn to perform the other operations of addition, subtraction, and multiplication. To do so requires a knowledge of *similar radicals*.

79. Similar Radicals are radicals which are of the same order, and have the same radicands. Such radicals may be added by merely combining their coefficients, the same as you would add $4\,x$, $5\,x$, and $6\,x$, making $15\,x$.

Illustrative examples:

I. $6\sqrt{3}+2\sqrt{3}-4\sqrt{3}=4\sqrt{3}$.

Radicals which do not appear to be similar are often found to be so after they have been properly simplified.

II. $2\sqrt{12}+5\sqrt{27}-\sqrt{48}+6\sqrt{\frac{1}{3}}$, when simplified by the rule of the preceding section, becomes

$$4\sqrt{3}+15\sqrt{3}-4\sqrt{3}+2\sqrt{3}=17\sqrt{3}.$$

Oral Exercises

Combine:

1. $3\sqrt{2} + 8\sqrt{2}$.
2. $4\sqrt{3} - 2\sqrt{3}$.
3. $6\sqrt{5} + 2\sqrt{5} - 3\sqrt{5}$.
4. $5\sqrt{6} + 7\sqrt{6} + \sqrt{6}$.
5. $3\sqrt{3} + 6\sqrt{3} - \frac{1}{2}\sqrt{3}$.
6. $\sqrt{12} + 4\sqrt{3}$.
7. $2\sqrt{18} + \sqrt{2}$.

Exercises

Simplify and combine similar terms:

1. $2\sqrt{3} + 6\sqrt{27} - 5\sqrt{48} - 2\sqrt{12}$.
2. $2\sqrt[3]{16} + 5\sqrt[3]{54} - \sqrt[3]{250} + \frac{1}{2}\sqrt[3]{128}$.
3. $\sqrt[3]{24\,a^2} + \sqrt[3]{81\,a^2} - 6\sqrt[3]{3\,a^2}$.
4. $x\sqrt{y} + \sqrt{x^2y} + 2\,x\sqrt{y}$.
5. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{8}{25}} + 3\sqrt{32}$.
6. $.04\sqrt{72} + .05\sqrt{98} - \frac{1}{5}\sqrt{50}$.
7. $\sqrt{2\,a} + 3\sqrt{8\,a^3} - \sqrt{98\,a^3} + 2\,a\sqrt{\frac{1}{2}\,a}$.
8. $3\sqrt{20} + 5\sqrt{\frac{1}{5}} - \frac{1}{3}\sqrt{45} - 2\sqrt{80}$.
9. $\sqrt{a^3} + \sqrt[4]{a^2} + 3\sqrt[6]{a^3}$.
10. $\sqrt[3]{\frac{x^3}{9}} + \sqrt[3]{\frac{y^6}{9}} + \sqrt[3]{\frac{z^9}{9}}$.
11. $\sqrt[6]{n} + \sqrt[3]{\frac{n}{64}}$.
12. $\frac{3}{5}\sqrt{75\,a^2} - \frac{48}{a^3}\sqrt{3\,a^8} - a^2\sqrt{\frac{12}{a^2}} + 36\,a\sqrt{5\frac{1}{3}}$.
13. $\sqrt[3]{-16\,x} + \sqrt[3]{-250\,x} - \sqrt[3]{-54\,x}$.
14. $\sqrt{(H+K)^3} + 3K\sqrt[8]{(H+K)^4} - 2K\sqrt[4]{(H+K)^2}$.
15. $\sqrt[3]{c^3d^2} + \sqrt[6]{\frac{1}{d}} - \sqrt[3]{64\,d^5}$.
16. $\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}$.

17. $\sqrt{2^2 \cdot 3} + \sqrt[4]{2^4 \cdot 3^2} - \sqrt[6]{2^6 \cdot 3^3}$.

18. $\frac{2d}{c^2 - d^2}\sqrt{c^2 - d^2} + \sqrt{\frac{c+d}{c-d}} - \sqrt{\frac{c-d}{c+d}}$.

19. $\sqrt{(a-b)^3} + 2\sqrt{(a^2 - b^2)(a+b)} + \sqrt{\frac{1}{a-b}}$.

20. $\sqrt{5x^2 - 10x + 5} - \sqrt{5x^2 + 20x + 20}$.

21. $\sqrt{4a - 8} + \sqrt{16a - 32}$.

22. $\sqrt{x+1} - \sqrt{(x+1)^3}$.

23. $\sqrt{(x+2)^3} - \sqrt{4x+8} - \sqrt{x^3 + 2x^2}$.

24. $\sqrt{4c^3 + 4c^2d} + \sqrt{4cd^2 + 4d^3}$.

25. $H\sqrt{H^3K^7} + K^2\sqrt{H^7K^3} - 2HK^2\sqrt{H^5K^3} + \sqrt[5]{H^{28}K^{28}}$.

26. $\sqrt{9x + 9y} + \sqrt{4x + 4y} + \sqrt{x+y}$.

27. $4x^{\frac{1}{2}} + 12x^{\frac{3}{2}} + 9x^{\frac{5}{2}}$.

Hint: First remove the common factor $x^{\frac{1}{2}}$, or its equivalent $\sqrt{x}$.

28. $4x^{\frac{1}{3}}y^{\frac{5}{3}} - 3x^{\frac{7}{3}}y^{\frac{3}{3}}$.

29. $x^{\frac{1}{2}} + 3x^{\frac{3}{2}} + 4x^{\frac{5}{2}}$.

30. $8^{\frac{1}{2}} + 32^{\frac{1}{2}} - 50^{\frac{1}{2}}$.

31. $3(27)^{\frac{1}{2}} - 4(12)^{\frac{1}{2}} + 5(75)^{\frac{1}{2}}$.

32. $2\sqrt{12} - 3(27)^{\frac{1}{2}} + \sqrt{75} - 2(108)^{\frac{1}{2}}$.

33. Find the value to the nearest hundredth of

$$12\sqrt{18} + 8\sqrt{2} - 5\sqrt{16}.$$

34. Find the sum of the lengths of the diagonals of three squares whose sides are 6 feet, 11 feet, and 14 feet, respectively.

Review

1. If 3 feet be added to both sides of a certain square, its area is increased by 81 square feet. Find the side of the original square.

2. Find the value of y in

$$.15\,y + \frac{.135\,y - .225}{.6} = \frac{.36}{.2} - \frac{.09\,y - .18}{.9}.$$

3. Divide $H^8 - 81\,K^4$ by $H^2 - 3\,K$.

4. How much water must be added to a quart of alcohol 95% pure, to make a solution 40% pure?

5. Express with positive exponents and find the values of:

(a) 2^{-3}. (d) $36 \cdot 3^{-2}$. (g) $9^0 \cdot 3^{-2} \cdot 27$.

(b) $8^{-\frac{2}{3}}$. (e) $9^{-\frac{1}{2}}$. (h) $32^{-\frac{1}{5}} \cdot 16^{\frac{1}{4}}$.

(c) $2^{-3} \cdot 3^{-2}$. (f) $27^{-\frac{1}{3}}$. (i) $17\,x^0$.

80. Multiplication of Radicals. — Radicals can be multiplied only when they are of the same order.

Rule. — *To multiply two radicals of the same order*

(1) *Multiply their coefficients for the coefficient of the product.*

(2) *Multiply the radicands to obtain the radicand of the product.*

(3) *Simplify the result.*

Illustrative examples:

I. $(-2\sqrt{3})(4\sqrt{6}) = -8\sqrt{18} = -24\sqrt{2}$.

II. $2\sqrt[3]{3} \cdot 6\sqrt[3]{9} = 12\sqrt[3]{27} = 36$.

III. $(3\sqrt{5})^2 = 9 \cdot 5 = 45$.

Notice that when a radical of the second order is squared the result is the radicand without the radical sign.

Thus $(\sqrt{2})^2 = (2^{\frac{1}{2}})^2 = 2$.

It is unnecessary to write this first as $\sqrt{4}$ and then simplify it, though this process gives the correct result. If, however, there is a coefficient to the radical, when the square is desired, you should not neglect to square this coefficient.

Thus $(-2\sqrt{3})^2 = 4 \cdot 3 = 12$.

Multiplication of polynomials containing radicals is sufficiently illustrated by the following example. You should notice that each partial product is simplified as soon as it is obtained, in order that like terms may be placed in vertical columns, for the purpose of addition.

Illustrative example:

Multiply $2\sqrt{6} - 3\sqrt{2} + \sqrt{3}$ by $2\sqrt{6} + 3\sqrt{2} - 2\sqrt{3}$.

Solution:

$$\begin{array}{rl} & 2\sqrt{6} - 3\sqrt{2} + \sqrt{3} \\ & 2\sqrt{6} + 3\sqrt{2} - 2\sqrt{3} \\ \hline 2\sqrt{6}(2\sqrt{6} - 3\sqrt{2} + \sqrt{3}) = & 24 - 12\sqrt{3} + 6\sqrt{2} \\ 3\sqrt{2}(2\sqrt{6} - 3\sqrt{2} + \sqrt{3}) = & -18 + 12\sqrt{3} + 3\sqrt{6} \\ 3\sqrt{2}(2\sqrt{6} - 3\sqrt{2} + \sqrt{3}) = & -6 - 12\sqrt{2} + 6\sqrt{6} \\ \hline \text{Product} = & -6\sqrt{2} + 9\sqrt{6} \end{array}$$

81. Change of Order. — To multiply radicals of different order it is necessary to change them to the same order. The method by which this is done is indicated in the following:

Illustrative examples:

Multiply $3\sqrt{5}$ by $-2\sqrt[3]{3}$.

Solution:

The radicals must be changed to sixth roots.

Thus $3\sqrt{5} = 3 \cdot 5^{\frac{1}{2}} = 3 \cdot 5^{\frac{3}{6}} = 3\sqrt[6]{5^3} = 3\sqrt[6]{125}$,

and $-2\sqrt[3]{3} = -2 \cdot 3^{\frac{1}{3}} = -2 \cdot 3^{\frac{2}{6}} = -2\sqrt[6]{3^2} = -2\sqrt[6]{9}$.

Multiplying $3\sqrt[6]{125}$ by $-2\sqrt[6]{9}$, we obtain $-6\sqrt[6]{1125}$.

The following is a statement of the

Rule. — *To multiply two radicals of different orders,*

(1) *Change them to that order whose index is the lowest common multiple of the given indices.*

(2) *Multiply as explained in the preceding rule, and simplify when possible.*

Oral Exercises

Find, mentally, the product of each of the following:

1. $\sqrt{2} \cdot \sqrt{4}$.
2. $\sqrt{20} \cdot \sqrt{5}$.
3. $\sqrt{a} \cdot \sqrt{a^5}$.
4. $\sqrt{3x} \cdot \sqrt{2x}$.
5. $-\sqrt{xz} \cdot \sqrt{xyz}$.
6. $\sqrt{\frac{1}{3}} \cdot \sqrt{\frac{3}{4}}$.
7. $\sqrt{\frac{5}{3}} \cdot \sqrt{15}$.
8. $\sqrt{\frac{a}{b}} \cdot \sqrt{\frac{b}{c}}$.
9. $\sqrt[3]{4} \cdot \sqrt[3]{2}$.
10. $2\sqrt{2} \cdot \sqrt{3}$.
11. $5\sqrt{3} \cdot \sqrt{5}$.
12. $6\sqrt{2} \cdot \sqrt{7}$.
13. $2\sqrt{5} \cdot 3\sqrt{3}$.
14. $2\sqrt{5} \cdot 8\sqrt{3}$.
15. $3\sqrt[3]{3} \cdot \sqrt[3]{9}$.
16. $\frac{1}{3}\sqrt{18} \cdot \sqrt{2}$.
17. $x\sqrt{y^3} \cdot x^2\sqrt{y}$.
18. $\sqrt{xy} \cdot \sqrt{yz} \cdot \sqrt{zx}$.
19. $\sqrt[3]{x^7} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{x^3}$.
20. $(2\sqrt{2})^2$.
21. $(3\sqrt{5})^2$.
22. $(4\sqrt{3})^2$.
23. $(5\sqrt{6})^2$.
24. $(2\sqrt{5})^2$.
25. $(3\sqrt{3})^2$.
26. $(\sqrt{a}+1)^2$.
27. $(2+\sqrt{b})^2$.
28. $(\sqrt{x}-4)^2$.
29. $(\sqrt{m}-1)^2$.
30. $(\sqrt{a}+\sqrt{b})^2$.
31. $(\sqrt{x}-\sqrt{y})^2$.
32. $(2\sqrt{x}+3\sqrt{y})^2$.
33. $(3\sqrt{a}-2\sqrt{b})^2$.
34. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.
35. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$.
36. $(\sqrt{2}+3)^2$.
37. $(\sqrt{a}+2)(\sqrt{a}+4)$.
38. $(\sqrt{x+6}+\sqrt{3})(\sqrt{x+6}-\sqrt{3})$.
39. $(\sqrt{x+9}-\sqrt{5})(\sqrt{x+9}+\sqrt{5})$.

40. If one of the factors of 7 is $\sqrt{x+12}+\sqrt{x+5}$ what is the other?

41. If one of the factors of 7 is $\sqrt{x+11}+\sqrt{x+4}$ what is the other?

42. If one of the factors of 7 is $\sqrt{x+10}-\sqrt{x+3}$ what is the other?

43. If one of the factors of 5 is $\sqrt{x+6}+\sqrt{x+1}$ what is the other?

44. If one of the factors of 5 is $\sqrt{x+12}-\sqrt{x+7}$ what is the other?

45. If one of the factors of 5 is $\sqrt{x+9}+\sqrt{x+4}$ what is the other?

46. Make up factors in a similar manner for each of the following: 3, 11, 13, 17.

47. $\sqrt{x}\,\sqrt[3]{x} = ?$

48. $\sqrt{2}\,\sqrt[3]{2} = ?$

Exercises

Perform the indicated operation in each of the following examples:

1. $(\sqrt{3})(2\sqrt{5})$.
2. $(2\sqrt{8})(3\sqrt{2})$.
3. $(-5\sqrt{2})(6\sqrt{32})$.
4. $(-3\sqrt{27})(-2\sqrt{3})$.
5. $(5\sqrt{5})(-2\sqrt{15})$.
6. $(-3\sqrt{2})(-\sqrt{50})$.
7. $(4\sqrt{6})^2$.
8. $(-3\sqrt{5})^2$.
9. $(2\sqrt[3]{2})(6\sqrt[3]{4})$.
10. $(4\sqrt[3]{2})(\sqrt[3]{16})$.
11. $(2\sqrt[3]{3})^2$.
12. $(2\sqrt[3]{3})^3$.
13. $(\sqrt[4]{2})(2\sqrt[4]{32})$.
14. $(3\sqrt{10})(-4\sqrt{20})$.
15. $(7\sqrt{6})(-5\sqrt{30})$.
16. $\sqrt{3}(2\sqrt{2}-\sqrt{3})$.
17. $3\sqrt{2}(4\sqrt{2}-1)$.
18. $5\sqrt{3}(2\sqrt{3}-3\sqrt{2})$.
19. $\sqrt{5}(5\sqrt{3}+3\sqrt{5})$.

20. $2\sqrt{6}(\sqrt{3} - 2\sqrt{6} + \sqrt{12})$.
21. $-\sqrt{3}(2\sqrt{6} + 3\sqrt{3} - 5\sqrt{18})$.
22. $3\sqrt{a}(\sqrt{a} + \sqrt{b})$.
23. $-2\sqrt{ab}(\sqrt{a} - \sqrt{b})$.
24. $(3 + \sqrt{2})(3 - \sqrt{2})$.
25. $(3\sqrt{2} - \sqrt{3})(3\sqrt{2} + \sqrt{3})$.
26. $(5\sqrt{6} + \sqrt{5})(5\sqrt{6} - \sqrt{5})$.
27. $(\sqrt{3} + 2\sqrt{2} - 1)^2$.
28. $(3\sqrt{2} + 2\sqrt{5} + \sqrt{3})^2$.
29. $(\sqrt{5} - \sqrt{3} + 1)(\sqrt{5} - \sqrt{3} - 1)$.
30. $(4\sqrt{3} + 2\sqrt{6} - 1)(4\sqrt{3} - 2\sqrt{6} + 1)$.
31. $(3\sqrt{7} - 2\sqrt{2} - \sqrt{3})(3\sqrt{7} - 5\sqrt{2} + 2\sqrt{3})$.
32. $(a + b + \sqrt{ab})(a + b - \sqrt{ab})$.
33. $(2\sqrt{a} - b)(a + 2\sqrt{b})$.
34. $(2 - 2\sqrt{x})^2$.
35. $(\sqrt{4 + \sqrt{25}})(\sqrt{16 - \sqrt{49}})$.
36. $\sqrt{3} \cdot \sqrt[3]{5}$.
37. $2\sqrt{5}(-3\sqrt[4]{6})$.
38. $5\sqrt[3]{2} \cdot 3\sqrt[4]{2}$.
39. $(-5\sqrt{6})(-2\sqrt[3]{3})$.
40. $5\sqrt[6]{5} \cdot 3\sqrt[3]{10}$.
41. $\sqrt{2} \cdot \sqrt[3]{4} \cdot \sqrt[6]{6}$.
42. $(-2\sqrt{3})(-3\sqrt[3]{4})(-4\sqrt[4]{2})$.
43. Which is greater $\sqrt[3]{11}$ or $\sqrt{5}$?

Hint: Change to radicals of same order.

44. $5\sqrt{2}$ or $3\sqrt[3]{10}$?

Review

1. Multiply:

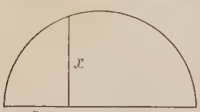
(a) $a^{\frac{1}{2}}a^{\frac{1}{3}}$. (b) $2^{1.5} \cdot 2^{1.5}$. (c) $10^{1.5} \cdot 10^{1.25} \cdot 10^{.25}$.

2. $(a + b)^2(a + b) = ?$

3. Define and give an example of

(a) A rational number.
(b) The radicand.
(c) A surd.
(d) A quadratic surd.
(e) An entire surd.
(f) A mixed surd.
(g) The coefficient of a radical.

4. The square root of a number may be found geometrically on squared or coördinate paper, by the following plan.



Let x be the square root to be obtained. Then from a fact of geometry and by the use of the principle of the mean proportional or by its equivalent, a certain fractional equation, we obtain the following in solving for the square root of 15.

$$\frac{5}{x} = \frac{x}{3} \text{ or, solving, we obtain } x = \sqrt{15}.$$

Note that the 5 and 3 are factors of 15. By measuring the line x in the figure we might obtain the square root of 5.

By a similar plan obtain

(a) $\sqrt{8}$. (b) $\sqrt{12}$. (c) $\sqrt{6}$.

82. Fifth Principle of Simplification. — We shall now illustrate the fifth principle of simplification. If we are asked to simplify the expression $\frac{\sqrt{2}}{\sqrt{5}}$, we may do so by multiplying both numerator and denominator by $\sqrt{5}$.

Thus $$\frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}.$$

One purpose of making the denominator rational is that its numerical value may be more easily computed.

Which is easier, to extract the square root of 10 and then divide by five, or to extract the square roots of both 2 and 5 and divide the former by the latter?

Then is $\frac{\sqrt{10}}{5}$ simpler or not simpler than $\frac{\sqrt{2}}{\sqrt{5}}$?

Illustrative examples.

I. $$\frac{3\sqrt{2}}{2\sqrt{3}} = \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}.$$

II. $$\frac{2\sqrt[3]{3}}{\sqrt[3]{2}} = \frac{2\sqrt[3]{3}}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{2\sqrt[3]{12}}{2} = \sqrt[3]{12}.$$

Rule. — *To simplify a fraction having an irrational denominator,*

(1) *Multiply both numerator and denominator of the fraction by the simplest expression which will produce a rational denominator.*

(2) *Simplify the result, if necessary, by the processes already explained.*

Radicals of different orders must first be reduced to radicals of the same order, by the use of fractional exponents.

Thus $$\frac{\sqrt{2}}{\sqrt[3]{2}} = \frac{2^{\frac{1}{2}}}{2^{\frac{1}{3}}} = \frac{2^{\frac{3}{6}}}{2^{\frac{2}{6}}} = \frac{\sqrt[6]{2^3}}{\sqrt[6]{2^2}}.$$

This may then be simplified by the above principle or better by the use of the principle stated in the second illustrative example of the next section.

83. Division of Radicals. — If the dividend and divisor are monomials of the same order, we may use the following principle,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Illustrative examples:

I. Divide $6\sqrt{6}$ by $-2\sqrt{3}$.

Solution: $$\frac{6\sqrt{6}}{-2\sqrt{3}} = -3\sqrt{2}.$$

Note that the coefficient of the denominator should be divided into the coefficient of the numerator whenever possible.

Illustrative example II:

Divide $3\sqrt{6}$ by $4\sqrt{5}$.

Solution: $$\frac{3\sqrt{6}}{4\sqrt{5}} = \frac{3\sqrt{6}}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{30}}{20}.$$

Illustrative example III:

Divide $8\sqrt{x}$ by $2\sqrt[3]{y}$.

Solution:

$$\frac{8\sqrt{x}}{2\sqrt[3]{y}} = 4\frac{\sqrt{x}}{\sqrt[3]{y}} = \frac{4x^{\frac{1}{2}}}{y^{\frac{1}{3}}} = \frac{4x^{\frac{3}{6}}}{y^{\frac{2}{6}}} = 4\sqrt[6]{\frac{x^3}{y^2}} = 4\sqrt[6]{\frac{x^3y^4}{y^6}} = \frac{4}{y}\sqrt[6]{x^3y^4}.$$

Note that these radicals are of different orders and are first reduced to the same order, and that the coefficient of the dividend is exactly divisible by the coefficient of the divisor.

Rule. — *To divide an algebraic expression by a monomial radical,*

(1) *All radicals should be of the same order or reduced to the same order.*

(2) *Indicate the division in fractional form.*

(3) *Divide the coefficient of the dividend by the coefficient of the divisor and the radicand of the dividend by the radicand of the divisor, wherever possible, and write the quotient of the radicands under the common radical sign.*

(4) *Simplify all radicands as explained in the above examples.*

(5) *If direct division is not possible, use the method of illustrative example II above.*

Oral Exercises

Simplify:

1. $\frac{\sqrt{12}}{\sqrt{2}}$.
2. $\frac{2\sqrt{6}}{\sqrt{3}}$.
3. $\frac{-3\sqrt{18}}{\sqrt{2}}$.
4. $\frac{-12\sqrt{24}}{2\sqrt{8}}$.
5. $\frac{10\sqrt{40}}{-2\sqrt{5}}$.
6. $\frac{16\sqrt{70}}{8\sqrt{5}}$.
7. $\frac{-12\sqrt{48}}{-4\sqrt{6}}$.
8. $\frac{10\sqrt[3]{81}}{2\sqrt[3]{9}}$.
9. $\frac{2\sqrt[4]{128}}{3\sqrt[4]{32}}$.
10. $\frac{6\sqrt{3}+3\sqrt{6}}{3\sqrt{3}}$.
11. $\frac{4\sqrt{10}-6\sqrt{15}}{-2\sqrt{5}}$.
12. $\frac{5\sqrt{6}-7\sqrt{3}}{2\sqrt{3}}$.

Exercises

Simplify:

1. $\frac{\sqrt{3}}{2\sqrt{6}}$.
2. $\frac{4\sqrt{2}}{2\sqrt{3}}$.
3. $\frac{5\sqrt{15}}{2\sqrt{5}}$.
4. $\frac{-6\sqrt{3}}{\sqrt{6}}$.
5. $\frac{3\sqrt{5}}{-\sqrt{10}}$.
6. $\frac{4\sqrt{2}}{3\sqrt{8}}$.
7. $\frac{7\sqrt{2}}{3\sqrt{3}}$.
8. $\frac{5\sqrt{12}}{3\sqrt{6}}$.
9. $\frac{8\sqrt{15}}{-4\sqrt{5}}$.

Divide:

10. $5\sqrt{30}$ by $8\sqrt{6}$.
11. $6\sqrt{2}$ by $2\sqrt{3}$.
12. $11\sqrt{12}$ by $2\sqrt{5}$.
13. 4 by $3\sqrt{6}$.
14. $3\sqrt{15}$ by $2\sqrt{60}$.
15. $3\sqrt[3]{12}$ by $2\sqrt[3]{3}$.
16. $\sqrt[4]{a^5}$ by $\sqrt[4]{a}$.
17. $\sqrt[3]{b}$ by $\sqrt{2}$.
18. $2\sqrt{50}$ by $\sqrt{\frac{1}{2}}$.
19. $\sqrt{\frac{1}{5}}$ by $\sqrt{\frac{4}{15}}$.

20. $\sqrt{.18}$ by $\sqrt{.02}$.

21. 2 by $\sqrt[4]{2}$.

22. $16\sqrt{a}$ by $8\sqrt[3]{b}$.

23. $4\sqrt{6}$ by $2\sqrt[3]{3}$.

24. $\sqrt[4]{8}$ by $\sqrt{2}$.

25. $\sqrt{\frac{1}{3}}$ by $\sqrt[3]{\frac{1}{3}}$.

Find the value of:

26. $3\sqrt{10} \div 2\sqrt{2}$.

27. $5\sqrt[3]{18} \div 3\sqrt[3]{2}$.

28. $2\sqrt[3]{2} \div 3\sqrt[3]{6}$.

29. $14 \div (-2\sqrt{7})$.

Given $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, and $\sqrt{7} = 2.646$, find the value to the nearest hundredth of:

30. $\dfrac{5\sqrt{15}}{2\sqrt{5}}$.

31. $\dfrac{11\sqrt{12}}{2\sqrt{7}}$.

32. $\dfrac{6\sqrt{3}}{2\sqrt{2}}$.

33. $\dfrac{7\sqrt{2}}{3\sqrt{6}}$.

34. $\dfrac{8\sqrt{5}}{4\sqrt{3}}$.

35. $\dfrac{3\sqrt{2}}{-3\sqrt{6}}$.

Simplify:

36. $\dfrac{5\sqrt{2}}{\sqrt{3}}$.

37. $\dfrac{4\sqrt[3]{6}}{-3\sqrt{2}}$.

38. $\dfrac{5\sqrt[4]{7}}{2\sqrt[3]{3}}$.

39. $(\sqrt{14} - 5\sqrt{6}) \div 3\sqrt{2}$.

40. $(\sqrt{x} + \sqrt{y}) \div \sqrt{xy}$.

41. $(\sqrt{15} + 4\sqrt{20} - 8\sqrt{50}) \div \sqrt{10}$.

42. $(6\sqrt[3]{16} - 10\sqrt[3]{32}) \div (-2\sqrt[3]{2})$.

You will note that in working with radicals we are not accustomed to considering the square root of a number as being both + and −, but instead we use only the sign which precedes the radical.

Thus $\sqrt{4}$ is 2, and $-\sqrt{9}$ is used as -3.

Also $2\sqrt{25} - 3\sqrt{16} - \sqrt{9} = 10 - 12 - 3 = -5$.

Review

1. Which is greater, $\sqrt[3]{4}$ or $\sqrt{3}$? How much greater? Indicate the difference by a minus sign.

2. State the five principles of simplification of radicals.

3. Construct a right triangle, each of whose legs is 1, on squared paper. Find the value of the diagonal by the use of the right triangle equation.

4. In a 30°–60° right triangle the side opposite the 30° angle is $\sqrt{2}$. Find the hypotenuse. Find the other leg of the triangle.

84. Radicals Having Binomial Surd Denominators. — An algebraic expression consisting of two terms, one or both of which are surds, is called a *binomial surd;* as,

$$3 + \sqrt{5}; \quad 2\sqrt{a} - 4\sqrt{b}.$$

Two binomial surds which are exactly alike except that their second terms have opposite signs, as $2\sqrt{3} + \sqrt{5}$ and $2\sqrt{3} - \sqrt{5}$, are called *conjugate binomial surds.* The product of such a pair of radical expressions is always rational. Thus

$$\begin{array}{r} 2\sqrt{3} + \sqrt{5} \\ 2\sqrt{3} - \sqrt{5} \\ \hline 12 + 2\sqrt{5} \\ \quad - 2\sqrt{5} - 5 \\ \hline 12 - 5 = 7. \end{array}$$

A fraction whose denominator is a binomial surd may be simplified by the following

Rule. — *Multiply both numerator and denominator by the conjugate of the denominator. The result will be an equivalent fraction with rational denominator.*

Illustrative example:

Simplify: $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+\sqrt{3}}$.

Solution: $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+\sqrt{3}} = \dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+\sqrt{3}} \cdot \dfrac{3\sqrt{2}-\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$.

Multiplying both numerator and denominator by the conjugate of the denominator, we obtain

$$\frac{12+3\sqrt{6}}{15}.$$

Simplifying, it equals $\dfrac{4+\sqrt{6}}{5}$.

NOTE. It is understood in the discussion above that the radicals are of the second order.

Exercises

Simplify:

1. $\dfrac{3+\sqrt{6}}{3-\sqrt{6}}$.
2. $\dfrac{4\sqrt{2}+1}{4\sqrt{2}-1}$.
3. $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$.
4. $\dfrac{2\sqrt{3}-\sqrt{5}}{2\sqrt{3}+\sqrt{5}}$.
5. $\dfrac{5\sqrt{2}+1}{5\sqrt{2}+3}$.
6. $\dfrac{2\sqrt{3}+\sqrt{5}}{2\sqrt{3}+4\sqrt{5}}$.
7. $\dfrac{5\sqrt{2}-\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$.
8. $\dfrac{2\sqrt{3}+4}{3\sqrt{3}-2}$.

Hint: Indicate the quotient in fractional form, and simplify as above:

9. Divide $4\sqrt{3}+2\sqrt{5}$ by $4\sqrt{3}-2\sqrt{5}$.
10. Divide $2\sqrt{6}-\sqrt{2}$ by $3\sqrt{6}+2\sqrt{2}$.
11. Divide $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.

12. Simplify: $\dfrac{3\sqrt{a}+2\sqrt{b}}{\sqrt{a}-2\sqrt{b}}$.

13. Rationalize the denominator of $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}$.

14. Simplify: $\dfrac{2\sqrt{a}-\sqrt{b}}{\sqrt{a}+3\sqrt{b}}$. Compare your result with the original if $a = 9$ and $b = 4$.

15. Simplify: $\dfrac{6\sqrt{x}-5\sqrt{y}}{2\sqrt{x}+3\sqrt{y}}$.

16. Find the value correct to three decimal places of $\dfrac{a}{a-1}$ when $a = \sqrt{2}$.

17. Simplify: $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$.

18. Reduce to a single fraction with a rational denominator: $\dfrac{4+\sqrt{3}}{4-\sqrt{3}}-\dfrac{4-\sqrt{3}}{4+\sqrt{3}}$.

Simplify:

19. $\dfrac{1}{\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{3}-\sqrt{5}}-\sqrt{3}$.

20. $\dfrac{1}{\sqrt{5}-\sqrt{7}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+2\sqrt{5}$.

21. $\dfrac{1}{\sqrt{11}-\sqrt{7}}-\dfrac{1}{\sqrt{11}+\sqrt{7}}-3\sqrt{7}$.

Review

1. Using the values of $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, given in section 83, example 30, find the value of $3\sqrt{72}+2\sqrt{75}-4\sqrt{48}$ correct to hundredths.

2. Write in the simplest form:

$$36^{\frac{1}{2}}-\sqrt{98}+64^{\frac{1}{6}}-6\sqrt{2}+\sqrt[3]{27}.$$

3. Find the value of $6\,a^0 + \frac{1}{2^{-2}} - 3^4 + \frac{16}{4^{-2}} + (5467)^0$.

4. The equation showing the relation between the diagonal and the side of any square is $d = s\sqrt{2}$. Find the side of a square whose diagonal is 8. Whose diagonal is 5.

5. Solve $\frac{x - \frac{5}{3}}{\frac{5}{3}(x - 1)} + \frac{\frac{7}{3} - x}{\frac{7}{3}(1 + x)} = \frac{1}{35\left(1 - \frac{1}{x^2}\right)} + \frac{1}{7}$.

85. Square Root of Binomial Surd Expressions. — We may find the square roots of some binomial surds by inspection. While the square of a binomial is usually a trinomial, there are exceptions. If we should square the binomial surd $(\sqrt{3} + \sqrt{2})$, we should obtain $3 + 2\sqrt{6} + 2$ or $5 + 2\sqrt{6}$.

In studying the square thus obtained, we find that the 5 is the sum of the 2 and the 3 which we find in the radicands of the binomial surd with which we began. The 6 (the new radicand) is the product of those same numbers 2 and 3. Notice also that the coefficient of the radical in the square is 2. By observing these facts, we may obtain the square root of some binomial surds by inspection.

Illustrative example:

Find the square root of $7 + \sqrt{40}$.

Solution: We must first change our radical so that its coefficient shall be 2. We do this and obtain

$$7 + 2\sqrt{10}.$$

We now must find two numbers such that their sum shall be 7 and their product shall be 10. Obviously those two numbers are 5 and 2. We now have

$$\sqrt{7 + \sqrt{40}} = \sqrt{7 + 2\sqrt{10}}$$
$$= \sqrt{5} + \sqrt{2}. \quad \textit{Answer.}$$

Check:

$(\sqrt{5} + \sqrt{2})^2 = 5 + 2\sqrt{10} + 2 = 7 + 2\sqrt{10} = 7 + \sqrt{40}.$

Rule. — *To find the positive square root of a binomial surd by inspection:*

(1) *Reduce the surd term so that its coefficient shall be 2.*

(2) *Find two numbers such that their product shall equal the radicand of the surd obtained in* (1) *and their sum shall equal the rational term.*

(3) *Indicate the square roots of each of these numbers, the larger first, and connect them with the sign of the surd term.*

(4) *Simplify.*

Exercises

Find the square roots of the following by inspection:

1. $5 + 2\sqrt{6}$.
2. $6 - 2\sqrt{5}$.
3. $7 - 2\sqrt{12}$.
4. $11 - \sqrt{72}$.
5. $8 - \sqrt{28}$.
6. $6 - \sqrt{32}$.
7. $12 - 6\sqrt{3}$.
8. $8 + \sqrt{60}$.
9. $9 + 3\sqrt{8}$.
10. $2a + 2\sqrt{a^2 - 16}$.
11. $18 + 4\sqrt{18}$.
12. $30 + 10\sqrt{5}$.
13. $\sqrt{14 - 4\sqrt{10}} = ?$
14. $\sqrt{6x + 2\sqrt{9x^2 - y^2}} = ?$

86. Factoring by the Use of Radicals. — In your earlier work in factoring you were not permitted to use radicals in factoring for the obvious reason that you did not at that time have a knowledge of radicals.

They may, however, be used to factor some expressions which in the usual sense are prime.

Illustrative example I:

Factor $x^2 - 3$.

Solution:

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

Illustrative example II:

Factor $5x^2 - 1$.

Solution:

$$5x^2 - 1 = (\sqrt{5}\,x + 1)(\sqrt{5}\,x - 1).$$

Illustrative example III:

Factor $x - 2$.

Solution:

$x - 2 = (\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})$ or $(x^{\frac{1}{2}} - 2^{\frac{1}{2}})(x^{\frac{1}{2}} + 2^{\frac{1}{2}})$.

Illustrative example IV:

Factor $x^3 + 6$.

Solution: $x^3 + 6 = (x + \sqrt[3]{6})(x^2 - x\sqrt[3]{6} + \sqrt[3]{36})$.

The use of the radicals, however, in factoring is not usual because it is possible to go on and factor the expression into a countless number of factors.

Illustrative example V:

Solve by factoring $x^2 - 32 = 0$.

Solution:

$$x^2 - 32 = 0.$$

Factoring, $(x + \sqrt{32})(x - \sqrt{32}) = 0.$

Or, $x + \sqrt{32} = 0.$ $x - \sqrt{32} = 0.$

$x = -\sqrt{32}$ $x = \sqrt{32}$

$= -4\sqrt{2}.$ $= 4\sqrt{2}.$

Check:

With $x = -4\sqrt{2}$. With $x = 4\sqrt{2}$.

$(-4\sqrt{2})^2 - 32 = 0.$ $(4\sqrt{2})^2 - 32 = 0.$

$32 - 32 = 0.$ $32 - 32 = 0.$

Exercises

Factor:

1. $x^2 - 7$.
2. $5x^2 - 3$.
3. $x^3 - 3$.
4. $y^3 + 4$.
5. $5x^2 - 10$.
6. $7x^2 - 21$.
7. $4x^3 - 16$.

Solve by factoring and check:

8. $x^2 - 8 = 0$.
9. $x^2 - 6 = 0$.
10. $3x^2 - 9 = 0$.
11. $4x^2 - 48 = 0$.
12. $x^2 - 4\sqrt{2}\,x - 24 = 0$.
13. $6x^2 - 5\sqrt{2}\,x - 8 = 0$.

Simplify:

14. $\dfrac{a-b}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} - \dfrac{a^{\frac{3}{2}} - b^{\frac{3}{2}}}{a-b}$.
15. $\dfrac{a-b}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} - \dfrac{a+b}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.
16. $\dfrac{a^{\frac{1}{2}}b^{\frac{1}{4}}}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} \cdot \dfrac{a-b}{a^{\frac{1}{4}} + b^{\frac{1}{4}}}$.
17. $\dfrac{a^{\frac{1}{2}} + 1}{a + a^{\frac{1}{2}} + 1} \div \dfrac{1}{a^{\frac{3}{2}} - 1}$.

Review

1. Factor:
   (a) $x^2 - 1 + 2x - 2x^3$.
   (b) $a^4 - 3a^2 + 9$.
   (c) $18a^2 - 27a + 4$.
   (d) $x^{2a} - 2x^a + 1$.
   (e) $2x^3 + x^2 - 2x - 1$.
   (f) $1 + x^7$.

2. Find the values of each of the following:
   $(2^{-1})^2$, $(3^2)^{-1}$, $(5^{-2})^{-2}$, $(4^{\frac{1}{2}})^{-2}$, $(5^{\frac{1}{3}})^6$, $(6^4)^{-\frac{1}{2}}$, $(7^{-2})^{-1}$, $9x^0$.

3. Express as equivalent surds of the same order:
   (a) $\sqrt{2}$ and $\sqrt[3]{5}$.
   (b) $\sqrt{3}$ and $\sqrt[3]{4}$.
   (c) $\sqrt{3}$ and $\sqrt[5]{15}$.
   (d) $\sqrt{a}$ and $\sqrt[3]{a^2}$.

4. Solve by factoring:
   (a) $x^2 - 2x + 1 = 0$.
   (b) $x^2 - 2x - 3 = 0$.
   (c) $x(x+7) = 18$.
   (d) $(x+11)(x-5) = x + 11$.

5. Solve for the value of x:

(a) $x^{\frac{1}{3}} = -3$. (c) $x^{\frac{1}{2}} = \frac{1}{a}$.

(b) $x^{\frac{1}{4}} = c^4$. (d) $2x^{\frac{1}{4}} = 5$.

6. What value must x have in each of the following to make them true equations?

(a) $2^x = 4$. (c) $3^x = 81$. (e) $2^{3x} = 64$.

(b) $2^x = 32$. (d) $4^x = 64$. (f) $2^{4x} = \frac{1}{2}$.

87. Formulas Containing Radicals. — Formulas often contain radicals when solved for certain letters. In the case of formulas only the positive value of the radical should be used.

Exercises

1. (a) Solve for V, $K = \frac{1}{2}mV^2$.
   (b) Compute V to the nearest hundredth if $K = 40$ and $m = 6$. If $K = 20$ and $m = 6$.
2. (a) Solve for t, $S = \frac{1}{2}gt^2$.
   (b) Compute to the nearest tenth the value of t if $S = 150$ and $g = 32$.
3. (a) Solve for r, $A = \pi r^2$.
   (b) Compute r if $A = 154$ and $\pi = \frac{22}{7}$.
4. (a) Solve for r, $S = 4\pi r^2$.
   (b) Find r to the nearest tenth if $S = 242$.
5. Solve for r, $V = \frac{4}{3}\pi r^3$.
6. (a) Solve for r, $V = \pi r^2 h$.
   (b) Compute the value of r to the nearest hundredth if $V = 160$ and $h = 8$.
7. (a) Solve for b, $a^2 + b^2 = c^2$.
   (b) Find the value of b if $c = 17$ and $a = 8$.

8. (a) Solve for b, $\frac{a}{b} = \frac{b}{c}$.

(b) Compute b if $a = 4$ and $c = 9$.

(c) Compute b to the nearest hundredth if $a = 6$ and $c = 12$.

9. (a) Solve for s, $d = s\sqrt{2}$.

(b) Find s (the side of a square) when the diagonal $(d) = 8.484$.

10. $\text{Area}_{\square} = b\sqrt{d^2 - b^2}$, where d = diagonal of the rectangle and b = the base of the rectangle. Find the area if $b = 3$ and $d = 6.7083 = 3\sqrt{5}$.

11. (a) $\text{Area}_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)}$

where a = one side of triangle,

b = another side of triangle,

c = third side of triangle,

$s = \frac{a+b+c}{2}$.

Illustrative example:

Find the area of a triangle whose sides are 15, 16, and 17.

Solution:

$$A_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{24 \cdot 9 \cdot 8 \cdot 7}$$
$$= \sqrt{2^3 \cdot 3 \cdot 3^2 \cdot 2^3 \cdot 7}$$
$$= \sqrt{2^6 \cdot 3^2 \cdot 3 \cdot 7}$$
$$= 2^3 \cdot 3\sqrt{21}$$
$$= 24\sqrt{21}$$
$$= 24(4.582)$$
$$= 109.968.$$

$a = 15$

$b = 16$

$c = 17$

$s = \frac{15+16+17}{2} = 24$

$s - a = 9$

$s - b = 8$

$s - c = 7$

Correct to the nearest hundredth, we would have 109.97.

(b) Find the area of a triangle whose sides are 6, 8, and 12.

(c) Find the area of a triangle each side of which is 10.

12. Using the equation $h = \frac{gt^2}{2}$, where

h = height in feet from which an object falls,
g = 32.16 feet per second each second,
t = time in seconds.

Find how long it will take an object to fall to the earth from a height of 700 feet.

13. A body B is h feet above the surface of the earth. The number of miles m at which it may be seen is denoted by the equation $m = \sqrt{\frac{3h}{2}}$. If a certain lighthouse is 200 feet above the sea level, what is the greatest distance from which it may be seen?

14. $C_0 = 2\sqrt{\pi A}$, where C_0 = the circumference of a circle and A = area of the circle. Find C_0 if $A = 36$.

15. (a) Solve $\pi d^2 = 4A$ for d.

(b) Find d when $A = \frac{176}{7}$ and $\pi = \frac{22}{7}$.

16. The volume of a frustum of a pyramid is expressed by the formula $V = \frac{1}{3}h(b + b' + \sqrt{bb'})$, where h = the altitude and b and b' are the bases. If $h = 3$ feet and the bases are squares of sides 10 inches and 8 inches respectively, compute the volume to the nearest tenth of a cubic inch.

17. The horse power of a certain gasoline engine is denoted by $H = \frac{1}{6}D^2N$, where D equals the diameter of a cylinder in inches and N equals the number of cylinders. Solve this formula for D. Compute the diameter of the cylinder of a 20 horse-power engine of 6 cylinders.

18. The time in seconds (t) for the complete vibration of a simple pendulum is given by the formula $t = \pi\sqrt{\frac{l}{32.2}}$, where l represents the length in feet of the pendulum and $\pi = 3.1416$. Find l if a pendulum vibrates in 2 seconds.

19. The velocity of a falling body, falling from rest, is obtained from the equation $v = \sqrt{2gs}$.

(a) Solve this for s (the space fallen in feet).

(b) Find s if $g = 32.2$ and $v = 128.8$ feet per second.

20. The kinetic energy of a flywheel rim is given by the formula $K = \frac{Wv^2}{2g}$, where W equals the weight in tons, $g = 32$, and v = the velocity of the rim (meaning the rim speed, or the speed at which the rim is revolving). Solve this equation for v. Compute v if the kinetic energy is 480 and $W = 5$ tons.

Exercises

1. Find the side of a square whose area is 27 square inches.

2. What is the radius of a circle whose area is 24.2 square feet?

Hint: The area of a circle equals πr^2, where $\pi = 3.1416$ and r represents the radius of the circle.

3. What is the diagonal of a square whose side is 6?

4. What is the diagonal (d) of a square whose side is s?

5. By means of the equation obtained in number 4, what is the diagonal of a square whose side is 8? 10? 15? 20?

6. Solve your answer obtained in example 4 for the value of s.

7. What is the side of a square whose diagonal is $\sqrt{50}$?

8. Draw a square (preferably on squared paper) 4 units (or quarter inches) on a side. Measure the diagonal, then find the diagonal by the use of example 4. Compare your results. Which method is simpler?

9. (a) What is an equilateral triangle? Draw one.

(b) If one side of an equilateral triangle is 6, find the altitude a.

10. Find the side s of an equilateral triangle whose altitude is a.

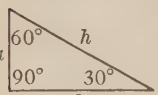
11. What is the side of an equilateral triangle whose altitude is 27?

12. If a horse is tied to a stake by a tether 30 feet long, over how many square feet may he graze?

Hint: See example 2 above.

13. The hypotenuse of a right triangle is 10. Each leg of the triangle is x. Find the value of x. Find the area of the triangle.

14. The hypotenuse of a right triangle, one of whose angles is 30° and the other 60° (called a 30°–60° triangle), is twice the length of the leg opposite the 30° angle. If this leg is $5\sqrt{2}$, what is the hypotenuse? What is the length of the other leg?



15. A rectangle is formed by placing side by side three equal squares each 6 inches on a side. Find its diagonal. Answer the same question if the side of each square is s.

16. A ladder 30 feet long is set with its foot 8 feet from a house. How high on the house will it reach?

17. The hypotenuse of an isosceles right triangle is 10 inches. How long is each leg?

18. If the gable of a house is 30 feet wide at the eaves and the ridge pole is 12 feet above level of the eaves, how long are the rafters?

19. A man walks 4 miles north, then three miles east, then two miles north. How far is he from his starting point? Prove your answer by the use of squared paper.

20. One leg of a right triangle is half the other and the hypotenuse has a length of $3\sqrt{5}$ feet. How long is each leg?

Review

1. What is the remainder when

$$6x^4 - 7x^3 + 5x^2 - 2x + 3$$

is divided by $x - 5$?

2. What must be added to each term of the fraction $\frac{m}{n}$ in order that the resulting fraction shall equal $\frac{h}{k}$?

3. Distinguish between $\frac{x+1}{2}$ and $\frac{x}{2} + 1$. Prove when $x = 5$.

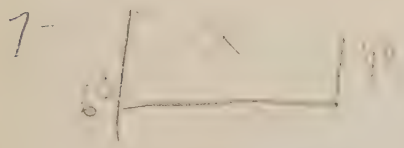
4. Express each of the following quotients as powers of fractions:

(a) $\frac{25}{81}$. (b) $\frac{8}{27}$. (c) $-\frac{1}{32}$. (d) $\frac{81}{625}$.

5. How much pure water must be added to a gallon of sulphuric acid, 96% pure, in order that the mixture shall be 80% pure?

6. Arrange the following numbers and find their median: 78%, 88%, 72%, 91%, 80%, 70%, 93%, 84%, 90%, 94%, 75%. Also find their average. How does the average differ from the median?

Hint: See Median, author's first course, page 67.

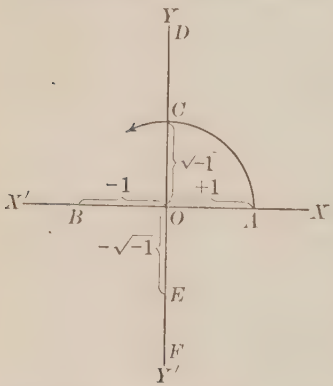


CHAPTER VIII

IMAGINARIES

Up to this time you have learned to use rational and irrational numbers. For the purpose of distinction we shall call these real numbers. In applying algebraic principles, however, we often meet a number which is neither rational nor irrational, but which is fundamentally different than either. We need to know something of this kind of number in order to fully understand algebraic processes.

88. Imaginary Numbers. — Since the square of either a positive number or of a negative number is always positive, it follows that no real number, when squared, can produce a negative number. The square root of a negative number is therefore neither positive nor negative.



An *imaginary number* is the indicated even root of a negative number; as $\sqrt{-2}$ and $2\sqrt{-9}$.

The *meaning* of an imaginary number is better understood by means of a diagram, as follows: Suppose that we wish to make clearer the meaning of $\sqrt{-1}$.

In your former work on graphs you learned to represent $+1$ as one unit to the right, measured on the x-axis, from the y-axis, and -1 as one unit to the left from the y-axis, as OA and OB, respectively, in the

figure. $OA = +1$ lies in the opposite direction to $OB = -1$. We may then think of OB as the result we would obtain if we were to rotate OA to the left around the point O through 180°. Now if we were to stop at OC, our rotation would then be through 90°. We shall think of this as the $\sqrt{-1}$. Now we shall make another *equal* rotation from OC to OB and we find that our result is -1.

Or to put this in another way, if we call $OA = 1$, then $OB = -1$. Then by rotating the $+1$ through 180°, we arrive at -1; but $-1 = +1(-1)$, so that our multiplier (-1) may be thought of as an operator which turns our representation of $+1$ through 180°, or two right angles in a direction opposite to the hands of a clock. Now since our multiplication of $+1$ by -1 turns our representation of $+1$ through two right angles, as above stated, we shall think of $\sqrt{-1}$ as an operator which turns our representation of $+1$ through one right angle in the direction opposite to the hands of a clock. Since there are two *equal* rotations to arrive at -1, then we have from the general law that the $(\sqrt[r]{a})^r = a$ the very important principle that $(\sqrt{-1})^2 = -1$.

We may also think of OD as $2\sqrt{-1}$, and since OE is in the opposite direction to OC, we may think of this as $-\sqrt{-1}$, and OF as $-2\sqrt{-1}$.

89. The Imaginary Unit. $-\sqrt{-1}$ is considered the unit of the imaginary system just as 1 is the unit of the system of real numbers. This means that every pure imaginary is a real number times $\sqrt{-1}$. The number $\sqrt{-1}$ is known to mathematicians as "i."

Thus
$$\sqrt{-9} = \sqrt{9 \cdot (-1)} = 3\sqrt{-1} \text{ or } 3i.$$
$$\sqrt{-6} = \sqrt{6}\sqrt{-1} = \sqrt{6}\,i.$$
$$-2\sqrt{-10} = -2\sqrt{10}\,i, \text{ etc.}$$

This, therefore, allows us to call OC in the preceding figure i, and OD, $2\,i$.

Similarly we call OE, $-\,i$, and OF, $-\,2\,i$.

The imaginary unit $\sqrt{-1}$ or i is used like the real terms or factors which you found in the beginning of your algebra study, and therefore you have another mathematical tool, which was unknown to the earlier mathematicians, to assist you in your future work.

Thus $4 = +1+1+1+1,$

and so $4\sqrt{-1} = \sqrt{-1}+\sqrt{-1}+\sqrt{-1}+\sqrt{-1},$

or $4\,i = i+i+i+i.$

$-4 = -1-1-1-1,$

and so $-4\sqrt{-1} = -\sqrt{-1}-\sqrt{-1}-\sqrt{-1}-\sqrt{-1},$

or $-4\,i = -i-i-i-i.$

$\sqrt{3}\cdot 6 = 6\sqrt{3},$

and so $\sqrt{-3}\cdot 6 = 6\sqrt{-3},$

or $\sqrt{3}\,i\cdot 6 = 6\sqrt{3}\,i.$

$$\frac{5\sqrt{6}}{\sqrt{6}} = 5,$$

and so $$\frac{n\sqrt{-1}}{\sqrt{-1}} = n,$$

or $$\frac{ni}{i} = n.$$

We then have two sets of imaginary numbers just as we had two sets of real numbers in the beginning of your algebra work, viz.:

$$\cdots -4\sqrt{-1},\ -3\sqrt{-1},\ -2\sqrt{-1},\ -1\sqrt{-1},\ 0,\ +1\sqrt{-1},\ +2\sqrt{-1},\ +3\sqrt{-1},\ +4\sqrt{-1},\ \cdots$$

or

$$\cdots -4\,i,\ -3\,i,\ -2\,i,\ -i,\ 0,\ +i,\ +2\,i,\ +3\,i,\ +4\,i,\ \cdots$$

The imaginary number is imaginary in the same sense that a negative number or an irrational number is imaginary for a person knowing only the positive integers.

We should be unable to solve such an equation as

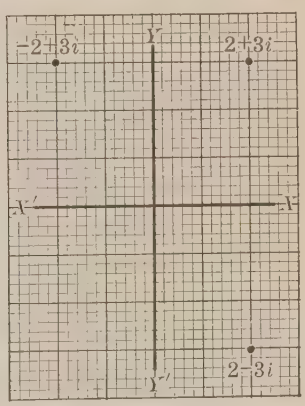
$$x^2 + 1 = 0$$

without a knowledge of imaginary numbers.

From this point we shall consider the Y–Y'-axis as the imaginary axis or the axis of imaginaries for work in imaginary numbers.

90. Complex Numbers. — A *complex number* is the indicated sum or difference of a real number and an imaginary number. Examples are, $3 + \sqrt{-1}$, $-2 + \sqrt{-6}$, and $2 - 5\sqrt{-1}$, or $3 + i$, $-2 + 6i$, and $2 - 5i$.

Obviously a complex number has one part of it which is always imaginary, though an imaginary number is not necessarily complex. Sometimes such expressions as $2\sqrt{-1}$ and $5\sqrt{-3}$ or $2i$ and $5i$ are called *pure imaginaries* as distinguished from complex numbers. We can obtain a better idea of complex numbers by the aid of the accompanying diagram. Suppose that we have the complex number $x = 2 + 3\sqrt{-1}$, and we wish to represent this on the diagram. This number is made up of the real number 2 and the imaginary $3\sqrt{-1}$. This means a distance of 2 units to the right, and then 3 units up. Similarly $x = -2 + 3\sqrt{-1}$ means 2 units to the left and then 3 units up. In the same way $x = 2 - 3\sqrt{-1}$ means 2 units to the right and then 3 units down.



A complex number may be represented in general form by $a + bi$, where a and b may have any real values whatever. When a has the value 0 the real term disappears, and the complex number becomes a pure imaginary. When b has the value 0 the imaginary term disappears, and the complex number becomes a real number. Thus both the real number and the pure imaginary may be considered as special cases of the complex number, which is, therefore, the most general form of number. For example, 6 may be considered as $6 + 0\,i$, and $4\,i$ may be considered as $0 + 4\,i$.

Exercises

Represent on a diagram the point x when

1. $x = 3 + i$.
2. $x = 3 - i$.
3. $x = -3 + i$.
4. $x = -3 - i$.
5. $x = 2 + 2\,i$.
6. $x = 3 - 2\,i$.
7. $x = -3 + 3\,i$.
8. $x = -2 - 3\,i$.
9. $x = -1 - 2\,i$.
10. $x = -3 + 2\,i$.
11. $x = 5 - 2\,i$.
12. $x = -4 + 3\,i$.
13. $x = 2 + 1.5\,i$.
14. $x = -3 + \frac{7}{2}\,i$.
15. $x = 2\,i + 3$.
16. $x = -3\,i - 3$.

17. Represent $4 + 2\,i$, $4 - 2\,i$. Also represent the sum of these numbers.

18. Represent $2 + i$, $2 - i$. What represents their sum? Where is this point?

19. The vertices of a triangle are $3 + 2\,i$, $-2 + 2\,i$, and $-1 - 3\,i$. Construct the triangle.

20. The vertices of a certain plane figure are $5 + 4\,i$, $-6 + 4\,i$, $5 - 3\,i$, and $-6 - 3\,i$. Plot these points. What is the figure? What is its area? What is the area of the figure whose vertices are the last three mentioned?

21. Represent $4 + 2\,i$, $4 - 2\,i$. Also represent the difference obtained by subtracting the latter from the former.

91. Simplification of Imaginaries. — Obviously an imaginary number is of necessity a radical in form, and many principles of simplification of radicals apply.

Principles 3 and 4 (page 165) are applied as in the simplification of real radicals.

Illustrative example I:

Simplify: $2\sqrt{-27} + 5\sqrt{-75} - 3\sqrt{-12}$.

Solution:

$$\begin{aligned} 2\sqrt{-27} + 5\sqrt{-75} - 3\sqrt{-12} &= 2\sqrt{9(-3)} + 5\sqrt{25(-3)} - 3\sqrt{4(-3)} \\ &= 6\sqrt{-3} + 25\sqrt{-3} - 6\sqrt{-3} \\ &= 25\sqrt{-3}. \end{aligned}$$

Illustrative example II:

Simplify: $\sqrt{-\frac{1}{2}} - 3\sqrt{-\frac{1}{18}} + 2\sqrt{-\frac{25}{8}}$.

Solution:

We obtain $\sqrt{-\frac{1}{2}\cdot\frac{2}{2}} - 3\sqrt{-\frac{1}{18}\cdot\frac{2}{2}} + 2\sqrt{-\frac{25}{8}\cdot\frac{2}{2}}$

$$\begin{aligned} &= \tfrac{1}{2}\sqrt{-2} - \tfrac{1}{2}\sqrt{-2} + \tfrac{5}{2}\sqrt{-2} \\ &= \tfrac{5}{2}\sqrt{-2}. \end{aligned}$$

Exercises

Simplify:

1. $2\sqrt{-8} - 3\sqrt{-8} + 4\sqrt{-72}$.
2. $\frac{1}{2}\sqrt{-9} + 6\sqrt{-4} + \sqrt{-25}$.
3. $4\sqrt{-20} - 6\sqrt{-80} + 2\sqrt{-45}$.
4. $a\sqrt{-a^2x} - 2b\sqrt{-b^2x} - 5c\sqrt{-c^2x}$.
5. $2\sqrt{-a^2 - b^2 + 2ab} + 3\sqrt{-6ab - a^2 - 9b^2}$.
6. $2\sqrt{-\frac{1}{2}} + \sqrt{-50} + 6\sqrt{-200} + 5\sqrt{-\frac{9}{2}}$.
7. $\sqrt{-12} + 4\sqrt{-75} - \sqrt{-\frac{100}{3}}$.
8. $3\sqrt{-x^2} + 4x\sqrt{-9} - \frac{2}{x}\sqrt{-25x^4}$.

92. Addition of Complex Numbers. — Complex numbers may be added as shown in the following:

Illustrative example I:

Add $2 + 5\sqrt{-1}$

$3 - 8\sqrt{-1}$

Sum $= 5 - 3\sqrt{-1}$

Illustrative example II:

Add $a + b\sqrt{-1}$

$c + d\sqrt{-1}$

Sum $= (a + c) + (b + d)\sqrt{-1}$

Illustrative example III:

Add: $4 + \sqrt{-9}$

$-6 + 2\sqrt{-4}$

Solution: $4 + \sqrt{-9} = 4 + 3\sqrt{-1}$

$-6 + 2\sqrt{-4} = -6 + 4\sqrt{-1}$

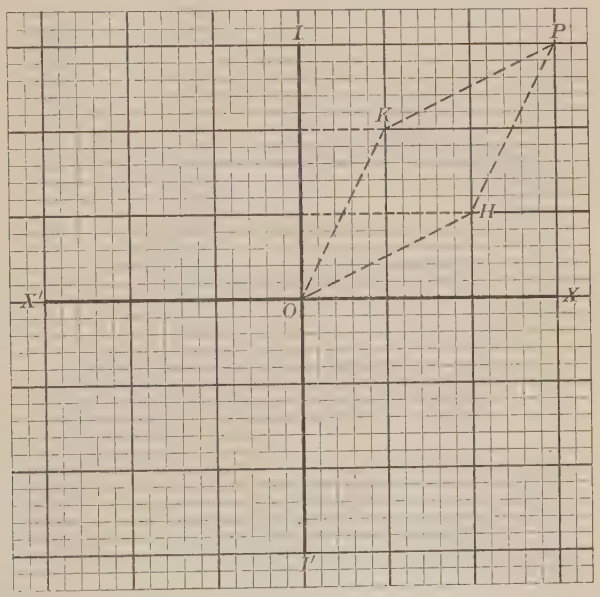
Sum $= -2 + 7\sqrt{-1}$

Exercises

Add:

1. $(3 + 2\sqrt{-1}) + (2 + 3\sqrt{-1})$.
2. $(2 + i) + (-3 + i)$.
3. $(1 + i) + (-2 + i)$.
4. $(1 + 2i) + (-3 + 3i)$.
5. $(-3 + 4i) + (-2 - 2i)$.
6. $(a + bi) + (c + 2di)$.
7. $(4 + \sqrt{-4}) + (-5 - \sqrt{-16})$.
8. $(5 - \sqrt{-4}) + (-4 - \sqrt{-9})$.
9. $(-2 + \sqrt{-3}) + (-2 - \sqrt{-3})$.
10. $(\frac{1}{4} - \frac{1}{4}i\sqrt{3}) + (3 + 3i\sqrt{3})$.
11. $(2\sqrt{2} - \sqrt{-12}) + (\sqrt{2} - \sqrt{-3})$.

93. Graphical Representation of the Addition of Two Complex Numbers. — The addition of two complex numbers may be shown graphically as illustrated by the following:



Illustrative example:

Add graphically $1 + 2\,i$ and $2 + i$.

Solution: First construct on the chart the point H, $(2 + i)$, then construct the point K, $(1 + 2\,i)$.

Construct OK and OH, then construct the parallelogram $OHPK$.

We have now located the point P. Point P represents $(1 + 2\,i) + (2 + i)$. The result is $3 + 3\,i$.

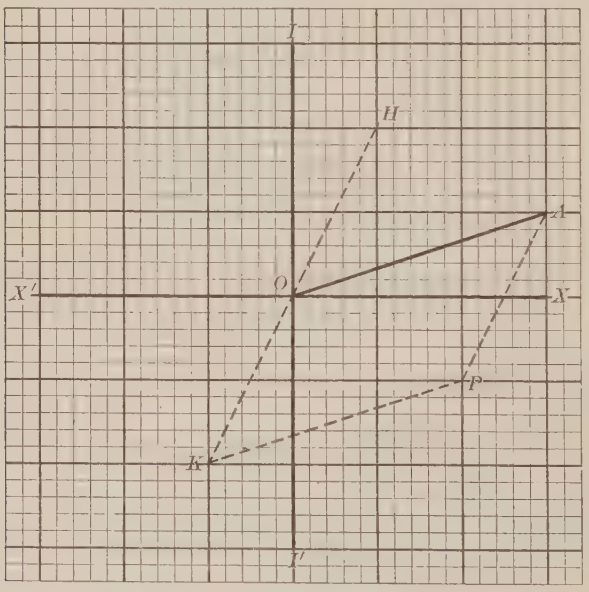
If we were to have drawn HP, equal and parallel to OK, we could have easily found the point P. This is due to a fact which you proved in your geometry.

Exercises

Represent graphically:

1. $(1 + 3\,i)+(3 + i)$.
2. $(1 + 2\,i)+(2 - 3\,i)$.
3. $(2 + 3\,i)+(3 - 2\,i)$.
4. $(3 + 2\,i)+(3 - 2\,i)$.
5. $(- 2 + i)+(+ 2 + i)$.
6. $(2)+(3 - i)$.
7. $(- 4\,i)+(3 + i)$.
8. $(4 - 2\,i)+(- 5\,i)$.

94. Graphical Representation of the Subtraction of Two Complex Numbers. — To subtract $1 + 2\,i$ from $3 + i$. This is equivalent to adding $- 1 - 2\,i$.



In order to do this graphically we find the point H, which represents $1 + 2\,i$, and the point K, which represents $- 1 - 2\,i$. Then we find the point A, which represents $3 + i$.

Karl Friedrich Gauss

Next we construct from A, a line equal and parallel to OK. The point P, shown in the adjoining figure, represents the result sought, that is, $2 - i$.

Exercises

Represent graphically:

1. $(1 + 3\,i) - (3 + i)$.
2. $(1 + 2\,i) - (2 - 3\,i)$.
3. $(2 + 3\,i) - (3 - 2\,i)$.
4. $(-4\,i) - (3 + i)$.

Other operations with complex numbers may be represented graphically, but these are found in the advanced algebra work.

HISTORICAL NOTE

Just as negative roots were not understood for many, many years, in the same way, the imaginary number and complex number were not understood by mathematicians for a long time. When imaginaries began to appear as roots of an equation, it was then, and then only, that some idea of their meaning began to be gained by the mathematical students.

They were not well understood until their graphical representation was shown by some writers. It was not until about 1800 that a Frenchman, Argand, and a Norwegian, Wessel, gave about the same representation of them. Karl Friedrich Gauss (1777–1855) in about 1800 published his demonstration that every algebraic equation has a root of the form of $a + bi$. He first put this subject upon a systematic and scientific basis.

95. Powers of i. — The table which follows gives the powers of i in succession.

$$i = \sqrt{-1}$$
$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = i^2 \cdot i = (\sqrt{-1})^2\sqrt{-1} = -1$$
$$i^4 = (i^2)^2 = (-1)^2 = +1$$
$$i^5 = i^4 \cdot i = \sqrt{-1}$$
$$i^6 = i^4 \cdot i^2 = 1, \text{ etc.}$$

You will observe that i^5 is the same, in value, as i, i^6 the same as i^2, etc. All powers of i reduce to one of the four quantities $\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$.

When the powers are arranged in order, they produce an indefinite number of cycles of these four quantities in the order given.

Thus $i^{46} = i^2 = -1$ (since $46 = 11 \cdot 4 + 2$).

And $i^{99} = i^3 = -\sqrt{-1}$ (since $99 = 24 \cdot 4 + 3$).

96. Multiplication of Imaginaries. — Let it be required to multiply $3\sqrt{-4}$ by $2\sqrt{-6}$.

Solution:
$$3\sqrt{-4} = 6\sqrt{-1}$$
$$2\sqrt{-6} = 2\sqrt{6}\sqrt{-1}$$
$$\text{The product} = 12\sqrt{6}(-1) = -12\sqrt{6}.$$

The product of two pure imaginary numbers is always real.

Again multiply $2 - 3\sqrt{-5}$ by $1 + 2\sqrt{-5}$.

Solution:
$$\begin{array}{l} 2 - 3\sqrt{5}\,i \\ 1 + 2\sqrt{5}\,i \\ \hline 2 - 3\sqrt{5}\,i \\ \quad + 4\sqrt{5}\,i - 30\,i^2 \\ \hline 2 + \sqrt{5}\,i - 30\,i^2 = 32 + \sqrt{5}\sqrt{-1} \end{array}$$

Conjugate complex numbers are those which differ only in the sign of the imaginary term, as $3 - 2\sqrt{-6}$ and $3 + 2\sqrt{-6}$.

The product of two conjugate complex numbers is real.

Thus
$$\begin{array}{l} 3 - 2\sqrt{6}\,i \\ 3 + 2\sqrt{6}\,i \\ \hline 9 - 6\sqrt{6}\,i \\ \quad + 6\sqrt{6}\,i - 24\,i^2 \\ \hline 9 - 24\,i^2 = 9 + 24 = 33 \end{array}$$

Note the similarity to $(a - b)(a + b)$.

97. Division of Imaginaries. — In dividing by an imaginary or complex number we indicate the division in fractional form, and simplify as in principle 5 of real radicals.

Illustrative example I:

Divide 3 by $2\sqrt{-3}$.

Solution:

$$\frac{3}{2\sqrt{-3}} = \frac{3}{2\sqrt{-3}} \cdot \frac{\sqrt{-3}}{\sqrt{-3}} = \frac{3\sqrt{-3}}{2(-3)} = \frac{3\sqrt{-3}}{-6} = -\frac{\sqrt{-3}}{2}.$$

Illustrative example II:

Simplify $\dfrac{3+\sqrt{-5}}{3-\sqrt{-5}}$.

Solution:

$$\frac{3+\sqrt{-5}}{3-\sqrt{-5}} = \frac{3+\sqrt{-5}}{3-\sqrt{-5}} \cdot \frac{3+\sqrt{-5}}{3+\sqrt{-5}} = \frac{4+6\sqrt{-5}}{14}$$
$$= \frac{2+3\sqrt{-5}}{7}.$$

Exercises

Perform the indicated operations and simplify the results:

1. $\dfrac{4}{3\sqrt{-1}}$.
2. $\dfrac{5}{i}$.
3. $\dfrac{1+i}{3i^3}$.
4. $\dfrac{2+3\sqrt{-2}}{2\sqrt{-2}}$.
5. $(2+5\sqrt{-1})(3-2\sqrt{-1})$.
6. $(2+3\sqrt{-1}-2\sqrt{-2})(1+3\sqrt{-1}+4\sqrt{-2})$.
7. $(2+\sqrt{-1}-\sqrt{-3})(2-\sqrt{-1}+\sqrt{-3})$.
8. $(1+i)^3$.
9. $(1-i)^3$.
10. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3$.
11. $\dfrac{2\sqrt{-1}}{3+2\sqrt{-1}}$.
12. $\dfrac{3\sqrt{-2}}{1-\sqrt{-1}}$.
13. $\dfrac{2+\sqrt{-3}}{2-\sqrt{-3}}$.

14. $\dfrac{3-\sqrt{-5}}{2+2\sqrt{-5}}$.

15. $\dfrac{-2+\sqrt{-4}}{-1+\sqrt{-4}}$.

16. $6\sqrt{-3}\cdot 4\sqrt{-3}$.

17. $(a+\sqrt{-b})(a-\sqrt{-b})$.

18. $(3-\sqrt{-1})(3+\sqrt{-1})$.

19. $(3-\sqrt{-2})^2$.

Divide:

20. $\sqrt{24}$ by $-\sqrt{-6}$.

21. $\sqrt{-12}$ by $\sqrt{-3}$.

22. $\sqrt{-45}$ by $2\sqrt{-9}$.

23. $\sqrt{-125}$ by $\sqrt{-25}$.

24. $(2+5\sqrt{-1})$ by $(2-5\sqrt{-1})$.

25. $(a-b\sqrt{-1})$ by $(a+b\sqrt{-1})$.

26. $(x+yi)$ by $(x+2\,yi)$.

27. Is $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ a root of $x^2+x+1=0$?

28. Simplify $(1-i)^3-2(1-i)^2+5(1-i)$.

29. If $x=\dfrac{1-i}{2}$, compute the value of x^2-3x+4.

Review — Word Problems

1. If $x+2$ is the larger part of $5x$, what is the smaller part?

2. If an airplane makes m miles an hour in still air, and the wind blows at the rate w miles an hour, what is the meaning of $m+w$? Of $m-w$? Of $\frac{3}{4}m+w$? Of $m+\frac{1}{2}w$?

3. The equation $\dfrac{x}{x+5}=\dfrac{2}{7}$ means that a fraction having a denominator 5 larger than the numerator equals $\frac{2}{7}$. What does each of the following mean?

(a) $\dfrac{x}{x+3}=\dfrac{1}{4}$.

(b) $\dfrac{x-5}{x}=\dfrac{3}{8}$.

(c) $\dfrac{x-15}{x}=\dfrac{1}{3}$.

4. If $\frac{x^2 + 4x + 4}{x^2 - 8x + 16}$ represents the length of a certain rectangle and $\frac{x - 4}{x + 2}$ represents the width, what is the area?

5. A cistern is filled by a pipe in m minutes. What part of the cistern will be filled by the pipe in 1 minute?

6. Into what number is 6 contained exactly x times?

7. A room is y yards long and f feet wide. How many square feet are there in the area of the floor?

8. A room is l feet long and w feet wide. What is the perimeter of the room?

9. How long will it take a man to ride m miles on a bicycle if he can ride 36 miles in h hours?

10. An automobile can go $(x^2 - xy - 3y^2)$ miles in $(x - 2y)$ hours. What is its speed of traveling? How far can it travel in $(x + y)$ hours? How many hours will it require to travel $(x + y)^3$ miles?

11. 96.4 is $16\frac{2}{3}\%$ more than what number? x is 20% more than what? c is 25% less than what?

12. A gain of \$3.75 is 25% of the cost. What was the cost? A gain of \$$g$ is 20% of the cost. What was the cost? A loss of c cents is $12\frac{1}{2}\%$ of the cost. What was the cost?

13. Goods costing \$9 are sold at a gain of 25% of the selling price. What was the selling price?

Goods costing \$$c$ are sold at a gain of 20% of the selling price. What was the selling price?

14. If a chicken loses 25% of its weight in dressing, what is the live weight of a chicken which dresses 1.5 pounds? What is the live weight of a chicken which dresses p pounds?

15. A framed picture is l inches long, and w inches wide. If the frame is x inches wide, what is the area of the part of the picture exposed to view?

16. If 5 pupils receive marks in a certain subject for the term as follows — $a\%$, $b\%$, $c\%$, $d\%$, $e\%$, what is the meaning of $\frac{(a+b+c+d+e)\%}{5}$? What is the name given to $c\%$ in this example?

17. A man has an automobile worth d dollars. It burns up and the insurance company pays him x dollars. How much is his loss?

18. The length of a rectangle is l. The length is four times the width. What is the area?

19. What is the average wage of m men at d dollars a day, n men at e dollars a day, and h men at f dollars a day?

20. Write the following word expressions with algebraic symbols, letting x represent the number of degrees in the angle.

(a) Three times the angle plus 15°.

(b) Five times the angle minus 12°.

(c) 23° less than the angle.

(d) Three times the angle minus 35°.

(e) Four times the result of adding 24° to the angle.

21. The difference between two numbers is d. The larger number is l. What is the smaller number?

22. A certain line is l inches long. Another line is i inches shorter. How long is the latter line? If the first represents the length of a rectangle and the latter its width, what is the area?

23. The word "mode" is usually described as meaning the *most common* thing. The word "average" refers to the mean, the quotient of any sum divided by the number of its terms. Find the mode in the following, then find the average:

Two men contributed \$20 each.

Five men contributed \$5 each.

Ten men contributed \$2 each.
Twenty-five men contributed \$1 each.
Eighty men contributed 50 cents each.
One hundred men contributed 25 cents each.

Which is the more common sized contribution, the average or the mode?

Review

1. Simplify:

$$4.8\,x^2 - 2\,x^2 - 6\,y^2 - 2.4\,xy + 2\,x^2 - (6.4\,xy - y^2) + 3.5\,y^2.$$

2. Factor:

(a) $9\,x^2 - 16$.
(b) $9\,x^2 - 4$.
(c) $x^4 + 4\,x^2 + 4$.
(d) $x^4 - 16\,x^2 + 64$.
(e) $(a - b)^2 - (a + b)$.
(f) $x^3 + 6\,x^2 + 11\,x + 6$.
(g) $x(x + 1)(x + 3) + x^2 + 4\,x + 3$.
(h) $x^2 + 3\,x - 28$.
(i) $x^4 + yx^3 + z^3x + z^3y$.

3. A square plot of ground would contain 73 square feet more if each side were 1 foot longer. Find the side of the square.

4. Simplify:

(a) $\dfrac{x}{9} + \dfrac{2}{3} + \dfrac{4}{x - 6} - \dfrac{2\,x}{3(x - 6)}$.

(b) $\dfrac{2}{a - x} - \dfrac{1}{2\,a - x} + \dfrac{1}{x}$.

5. Does $x = 7$ or 2 check in the following equation? Prove.

$$\sqrt{x + 7 + \sqrt{5(x - 2)}} = 3.$$

6. Find the square root of $4\,x^8 - 4\,x^6 - 7\,x^4 + 4\,x^2 + 4$.

7. Multiply $x^{-\frac{9}{4}} + 2\,x^{-\frac{3}{2}} + 4\,x^{-\frac{3}{4}} + 8$ by $x^{-\frac{3}{4}} - 2$.

8. What is the simplified form of $\dfrac{x^{-1} - y^0}{x^{-2} - y^0 x^{-1}} \cdot x^{-2}$?

9. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}})^2 = ?$

10. Solve:

(a) $b = 1.155(f - .2)$ for the value of f.

(b) $R = 3 + \frac{V}{6}$ for V.

(c) $R = \frac{gs}{g + s}$ for s. For g.

11. Simplify: $\frac{a^{x-2}b^{x+1}c^{x}}{x^{a-b}y^{c}z^{a-1}} \cdot \frac{x^{a+b}y^{b}z^{3}}{a^{2}b^{x+1}c^{3}}$.

12. If c represents the cost of a articles and p the price of one article, translate into words, $c = ap$.

13. Translate into words $a^2 + b^2 = c^2$.

14. If $A = p(1 + tr)$, where A = amount at simple interest, p = principal, t = time in years, r = rate, find the amount of \$640 at 5% for 4 years 3 months.

15. Solve the formula in example 14 for p. Then find the principal when the amount is \$350, the rate is 4%, and the time is 2 years 6 months.

16. If the population of a city in 1910 was 100,000 and in 1920 it was 110,000, what is the rate per cent of increase? Substitute x for the population in 1910 and y for the population in 1920, and write the per cent of increase. Suppose that it had decreased from x to z during the ten years, what would represent the per cent of decrease?

17. Verify the fact that 2 is a root of the equation $3x(x - 2) = x^2 - 4$. Is 2 a root of the equation obtained by dividing each member of the given equation by $x - 2$? Explain why it is or is not.

18. The length of a rectangle is l feet and the width is w feet. The length of another rectangle is $2l$ feet and its width is $2w$ feet. Find the ratio of the area of the first rectangle to the second rectangle.

19. The number of years in which a sum of money will double itself at the various rates of interest (compound interest, compounded annually) is shown in the following table.

Rate	Time	Rate	Time
4%	17.68 years	7%	10.25 years
$4\frac{1}{2}$%	15.76 years	8%	9 years
5%	14.21 years	9%	8.04 years
6%	11.91 years	10%	7.3 years

Draw bar graphs to represent these facts. What do you observe concerning the 4% and the 8% bars?

20. The following table shows the cost of \$1000 worth of life insurance in an insurance company at the different ages shown.

Age of Insured	20-Payment Life Policy
20	\$27.82
25	30.12
30	32.87
35	36.22
40	40.38
45	45.73
50	52.87

Draw bar graphs to illustrate these figures.

21. The following is the per cent of games won by the various baseball clubs of the National League in a certain year. Chicago, 65.1%; New York, 57.3%; Cincinnati, 53.1%; Pittsburgh, 52%; Brooklyn, 45.2%; Philadelphia, 44.7%; Boston, 42.7%; St. Louis, 39.5%. Show these relations with bar graphs.

22. The cost per capita (per person) for public schools in a recent year was reported as follows:

Pasadena, Cal.	$19.29	Cambridge, Mass.	$10.99
Yonkers, N. Y.	15.23	Jersey City	10.81
Los Angeles, Cal.	14.82	Bridgeport	10.29
Buffalo, N. Y.	13.03	St. Louis	9.40
Salt Lake City	12.45	Chicago	8.88
New York	12.32	Indianapolis	8.43
Washington	11.87	San Francisco	8.20
Pittsburgh	11.82	Philadelphia	7.73
Minneapolis	11.58	Birmingham, Ala.	4.98
Rochester	11.16	New Orleans	4.72
Cincinnati	11.01		

Construct bar graphs to represent the above facts. What was the median?

23. Given $\sqrt{2} = 1.414^+$ and $\sqrt{3} = 1.732^+$, compute in the shortest way, and correct to the nearest hundredth, the value of:

(a) $\frac{6}{\sqrt{3}}$. (b) $\frac{3}{\sqrt{2}}$. (c) $2\sqrt{17} \cdot \sqrt{34}$. (d) $3\sqrt{8} + 5\sqrt{18}$.

24. Evaluate:

(a) $16^{-\frac{1}{2}} \cdot 8$. (c) $16^{-1} \cdot 4$.

(b) $16^0 \cdot .7$. (d) $16^{-\frac{1}{2}}(4^0 + 9^{\frac{1}{2}})$.

25. State a power of a which, when divided by itself, equals a^0.

26. What is the difference between $(-2)^2$ and 2^{-2}?

27. What is the difference between $(a^2)^{-2}$ and $(a^{-2})^2$?

28. Write the coefficient and the exponent in each of the following:

(a) $2x^{-3}$. (b) $3x^{-\frac{1}{2}}$. (c) $5x^0$. (d) $.5x^6$. (e) $\frac{1}{2}x^{-\frac{1}{3}}$.

29. Write as a decimal fraction:

(a) 10^{-3}. (b) $5 \cdot 10^{-4}$. (c) $10^5 \cdot 10^{-9}$. (d) $10^5 \div 10^{10}$.

30. Solve for x and check:

$$1.7(x-2)-.3(2x+1)=.8.$$

31. Concrete is made of a mixture of cement, sand, and gravel. If the weights of sand and gravel are equal, but the weight of the cement used is three fifths of that of the sand, how many pounds of each are required to make 1450 pounds of concrete?

32. Solve the equation

$$\frac{x+b}{2a}+\frac{2a}{x-b}=\frac{x+a}{2a}.$$

33. Reduce to lowest terms

$$\frac{x^2-(y-z)^2}{(x-y)^2-z^2}.$$

33. Evaluate $1+\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}}$ when $a=2$, $b=3$, and $c=.5$.

35. Solve the following equations by factoring:

(a) $x^2+9x-70=0$. (e) $8x^2+24x-32=0$.
(b) $x^2-21x=100$. (f) $7x^2+98x-840=0$.
(c) $x^2-156=-x$. (g) $x^2+\frac{2}{3}x=87$.
(d) $x^2-53x=-150$. (h) $(x-2)^2+(x+5)^2=(x+7)^2$.

36. A and B together can do a piece of work in $3\frac{2}{3}$ days. How many days would it take each alone to do the work if B works twice as fast as A?

37. Multiply

(a) $\sqrt[3]{18}\cdot\sqrt{12}$. (b) $3\sqrt{2}-\sqrt{3}$ by $4\sqrt{2}+3\sqrt{3}$.

38. Combine $\dfrac{1}{x^2-1}+\dfrac{2x}{1-x}-\dfrac{3-x}{1+x}$.

39. Explain the difference between

(a) $-8^{\frac{2}{3}}$ and $(-8)^{\frac{2}{3}}$; (b) $5a^\circ$ and $(5a)^\circ$.

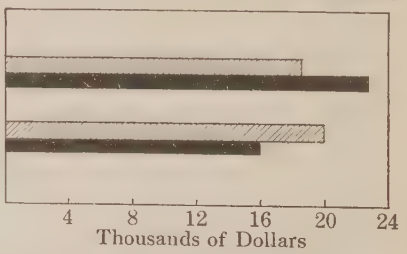
Exercises

1. Two factors concerning a subject are sometimes shown by the *vertical bar graph* separated into two parts, one dark and the other shaded as shown in the adjoining figure, from Brinton's "Graphical Representation of Facts."

Two factors may also be shown by what is known as the *dual bar graph* method shown in the lower figure. This is an adaptation of the above graph. In the vertical graph, the fire losses are shown in black, and the new buildings by the lightly shaded part of the bar. Each kind should be made to a scale and the scale properly marked on the axis. The idea of such graphs is to show two factors for the same year, or to show two factors for corresponding periods of time.

The lower graph shows the sales of two salesmen for the two successive years.

What relations do you observe from each of the adjoining graphs?



2. Three salesmen's sales in a certain month were \$1300, \$1100, and \$1500. Their sales the following month were \$1600, \$1800, and \$2000. Graph these facts by the use of the double bar graph.

What could a salesmanager learn from this graph if it were continued for many months?

3. Graph the following information:

	Daily Wages in 1899	Daily Wages in 1920
Trackmen	$1.00	$3.88
Station agent	1.75	6.96
Trainmen	1.95	6.40
Conductors	3.15	7.00

4. The expenses of two railroads for six successive years were:

	Ten Thousands					
	1915	1916	1917	1918	1919	1920
1st	26	44	46	50	64	68
2d	32	48	50	54	67	72

Graph these facts by the use of six vertical double bar charts. Connect the tops of the bars with a smooth curve.

CHAPTER IX

QUADRATIC EQUATIONS

HAVING completed our study of simple equations both integral and fractional, we are now ready to consider equations of the second degree in one unknown quantity. Equations containing the square of the unknown occur frequently in practice, and skill in solving them is necessary.

98. The Quadratic Equation has already been solved in this book under applications of factoring. The examples which follow are placed here as review and for the sake of completeness. If you find difficulty in solving them, refer to the previous discussion on page 83.

Exercises

1. Define a quadratic equation. Give several examples.

2. State the rule for solving a quadratic equation by the method of factoring.

3. Why do you have difficulty in attempting to solve the equation $x^2 + 4x + 5 = 0$ by factoring?

Solve by the use of factoring:

4. $x^2 - 8x - 20 = 0$.
5. $x^2 + 2x - 8 = 0$.
6. $x^2 - 8x + 7 = 0$.
7. $2x^2 + 5x = 3$.
8. $6x^2 - 7x = 5$.
9. $2x^2 - 7x = -6$.
10. $3x^2 - 6 = 17x$.
11. $2x^2 = 9 + 3x$.
12. $35 - 12x = -x^2$.
13. $2x^2 = -7x$.
14. $6x^2 - 23x - 18 = 0$.
15. $x^4 - 13x^2 + 36 = 0$.

16. $x^4 - 5x^2 + 4 = 0$.
17. $x^3 - 4x^2 + x + 6 = 0$.
18. $x^3 + 2x^2 - 5x - 6 = 0$.
19. $x^3 - 8 = 0$.
20. $x^4 = 8x$.
21. $(x^2 - 9) = 2(x + 3)$.
22. $0 = x^2 - 2x - 63$.
23. $x^2 = 5ax + 6a^2$.

24. What must be added as a third term to $x^2 + 6x$ in order that the three terms may be a perfect square?

In like manner complete each of the following so as to form perfect trinomial squares:

25. $x^2 - 12x$.
26. $x^2 + 2x$.
27. $x^2 + 20x$.
28. $x^2 + 4x$.
29. $x^2 - 18x$.
30. $x^2 - 30x$.
31. $x^2 + x$.
32. $x^2 + (?) + 16$.
33. $4x^2 + (?) + 9$.
34. $x^2 - (?) + \frac{1}{4}$.
35. $25x^2 + 60x$.
36. $\frac{K^2}{4} + 3K$.
37. $(?) + 12x + 36$.

Review

1. Prove that $a^m \cdot a^n = a^{m+n}$ when m and n are positive integers.

2. Find the value to the nearest tenth of $\frac{\sqrt{a} - \sqrt{x}}{\sqrt{a} + \sqrt{x}}$ when $x = 2$ and $a = 5$.

3. Does $x = 3 + \sqrt{-5}$ satisfy the equation $x^2 - 6x + 14 = 0$? Prove.

4. Arrange in descending order of magnitude $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{7}$. Show the work to prove.

5. (a) Solve for the value of t:

$$s = \tfrac{1}{2} gt^2.$$

(b) Find the value of t if $s = 700$ feet and $g = 32.16$ correct to the nearest tenth.

99. Completing the Square. — Since some quadratic equations cannot be factored, this method must be supplemented by others which may be used in any example. A fundamental method of this kind is that called completing the square. This is accomplished by the following

Rule. — (1) *Remove parentheses, clear the equation of fractions, and combine similar terms.*

(2) *Reduce the equation to the form* $x^2 + px = q$.

(3) *Complete the square by adding to both members of the equation the square of one half of the coefficient of* x.

(4) *Extract the square root of both members of the equation, prefixing the* $\pm$ *sign to the right member.*

(5) *Solve the resulting equations.*

Illustrative example I:

Solve $\dfrac{7}{x} + \dfrac{2x}{3} = \dfrac{13}{3}$.

Solution:

Clearing of fractions, $21 + 2x^2 = 13x$.

Arranging terms, $2x^2 - 13x = -21$.

Dividing by 2, $x^2 - \dfrac{13x}{2} = -\dfrac{21}{2}$.

Completing the square, $x^2 - \dfrac{13x}{2} + \dfrac{169}{16} = \dfrac{1}{16}$.

Extracting the square root, $x - \frac{13}{4} = \pm\frac{1}{4}$.

Adding $\frac{13}{4}$ to both members, $x = \frac{13}{4} \pm \frac{1}{4}$.

Combining similar terms, $= \frac{7}{2}$ or 3, roots

Illustrative example II:

Solve $\dfrac{x+1}{x^2} = \dfrac{a+1}{a^2}$.

Solution:

Clearing of fractions, $a^2x + a^2 = x^2(a+1)$.

Arranging, $(a+1)x^2 - a^2x = a^2$.

Dividing by $(a+1)$, $x^2 - \frac{a^2}{a+1}x = \frac{a^2}{a+1}$.

Completing square,

$$x^2 - \frac{a^2}{a+1}x + \frac{a^4}{4(a+1)^2} = \frac{a^4}{4(a+1)^2} + \frac{a^2}{a+1}.$$

$$= \frac{a^4 + 4a^3 + 4a^2}{4(a+1)^2}.$$

Extracting square root,

$$x - \frac{a^2}{2(a+1)} = \pm \frac{a(a+2)}{2(a+1)}.$$

Hence

$$x = \frac{a^2 \pm a(a+2)}{2(a+1)}$$

$$= a \text{ or } -\frac{a}{a+1}.$$

Exercises

Solve each of the following equations by the method of completing the square, and check as directed by the teacher:

1. $x^2 + 4x = 21$.
2. $x^2 - 10x = 11$.
3. $x^2 - 2x = 35$.
4. $x^2 + 8x = 33$.
5. $x^2 + 12x = 28$.
6. $x^2 + 5x = \frac{11}{4}$.
7. $x^2 + 3x = 4$.
8. $x^2 - 6x = 7$.
9. $x^2 + 12 = 8x$.
10. $12x^2 - 7x - 10 = 0$.
11. $3x^2 - 6x - 5 = 0$.

NOTE. The roots of this equation are *irrational*. Such roots should be simplified as explained in the chapter on radicals.

12. $x^2 + 10 = 8x$.
13. $x(x+4) = 7$.
14. $9x^2 - 12x - 1 = 0$.
15. $4x^2 - 11x = 45$.
16. $(1 - 3x)(x - 6) = 5(x + 2)$.
17. $(2x - 5)^2 - (x - 6)^2 = 80$.
18. $3(x^2 - x) = 2x^2 + 5x + 4$.
19. $5x^2 - 14x = -8$.
20. $x^2 - \frac{7}{12}x - 1 = 0$.
21. $x + \frac{1}{x} = \frac{17}{4}$.

22. $\frac{3x}{2} - \frac{1}{3x} = -\frac{1}{2}$.

23. $\frac{x}{3} + 4 = \frac{15}{x}$.

24. $(3x - 2)(x - 3) = x^2 + 5x + 4$.

25. $\frac{x-3}{x} = \frac{2x}{x-2}$.

27. $\frac{2}{x-2} - \frac{3x}{2} = 0$.

26. $\frac{2}{x-2} + 1\frac{7}{8} = \frac{1}{x+2}$.

28. $\frac{5-x}{12} - \frac{x}{x-5} = \frac{3}{2}$.

29. $ax^2 + bx + c = 0$.

Solution:

$$ax^2 + bx = -c.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

30. $x^2 - 8mx + 7m^2 = 0$.

31. $x^2 - 3bx = 10b^2$.

32. $x^2 + kx - 2k^2 = 0$.

33. $ax^2 + 2acx = m^2$.

34. $cx^2 + 4lx = -l^2$.

35. $10x^2 - 3b^2 = -bx$.

Solve the following equations, obtaining the roots correct to the nearest tenth:

36. $x^2 - 4x + 2 = 0$.

37. $3x^2 - 6x - 5 = 0$.

38. $4x^2 - 8x = 3$.

39. $\frac{x-3}{2} - \frac{x-2}{3x} = \frac{1}{x}$.

100. Solution by Formula. — The quadratic equation $ax^2 + bx + c = 0$ may be considered as a general type of any quadratic equation. Thus, if $a = 2$, $b = 5$, and $c = -6$, the equation would become $2x^2 + 5x - 6 = 0$; while if $a = 3$, $b = -1$, and $c = 2$, the equation then becomes $3x^2 - x + 2 = 0$, or if $a = 4$, $b = 8$, and $c = 0$, the equation then is $4x^2 + 8x = 0$. If $a = 5$, $b = 0$, $c = 0$, we have $5x^2 = 0$.

Thus, by giving a, b, and c every possible value, the equation would become in succession every possible quadratic equation. For this reason we say that the equation $ax^2 + bx + c = 0$ represents every quadratic equation.

In the last exercise, example 29, this equation was solved and the roots $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ obtained. Since the equation represents every quadratic equation, the roots just stated must represent the roots of every quadratic equation. Hence, in any given case we have only to substitute in the above expression as a formula the values of a, b, and c as they are in that particular equation and we shall have the roots of that equation.

Illustrative example:

Solve $2x^2 + x = 6$.

Solution: Transpose every term to the left member, as in the equation $ax^2 + bx + c = 0$.

We obtain $2x^2 + x - 6 = 0$.

Then a (the coefficient of x^2) $= 2$.
b (the coefficient of x) $= 1$.
c (the term without x) $= -6$.

Substituting these values in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we obtain $$x = \frac{-1 \pm \sqrt{(1)^2 - 4 \cdot 2(-6)}}{4}$$

$$= \frac{-1 \pm \sqrt{1 + 48}}{4}$$

$$= \frac{-1 \pm 7}{4}$$

$$= \tfrac{3}{2} \text{ or } -2.$$

Check with $\frac{3}{2}$:

$2(\frac{3}{2})^2 + \frac{3}{2} = 6.$
$2 \cdot \frac{9}{4} + \frac{3}{2} = 6.$
$\frac{9}{2} + \frac{3}{2} = 6.$
$6 = 6.$

Check with -2:

$2(-2)^2 + (-2) = 6.$
$2 \cdot 4 - 2 = 6.$
$8 - 2 = 6.$
$6 = 6.$

You should thoroughly memorize the quadratic formula as given above. By its use *any* quadratic equation may be solved, and this method will often be found easier and more rapid than the method of completing the square.

Exercises

By the use of the quadratic formula solve each of the following equations:

1. $x^2 - 8x + 15 = 0.$
2. $x^2 + 7x - 8 = 0.$
3. $x^2 + x = 42.$
4. $x^2 + 30 = 11x.$
5. $2x^2 - x = 1.$
6. $6x^2 - x - 15 = 0.$
7. $4x^2 - 23x = 6.$
8. $15x^2 - 16x = 15.$
9. $3x^2 + 7x - 2 = 0.$
10. $4x^2 = x + 8.$
11. $(x + 3)^2 = 10x + 5.$
12. $(5x - 1)(x + 6) = -25 + 5x.$
13. $(x + 7)(2 - x) = x^2 - 11.$
14. $(x - 3)^2 = (2x)^2 - 10x - 6.$
15. $(x - 2)(x - 3) = (2x - 3)^2 - 23.$

16. $2a^2x^2 - 7abx - 4b^2 = 0.$

Hint: In the formula, $a = 2a^2$, $b = -7ab$, $c = -4b^2$.

17. $5x^2 - 16ax + 3a^2 = 0.$

18. $b^2x^2 - 8bcx = -7c^2.$

19. $m^2x^2 - mnx - 12n^2 = 0.$

20. $x^2 - (a+b)x + ab = 0.$

21. $x^2 + (5b - 2a)x - 10ab = 0.$

22. $\frac{x}{2} + \frac{12}{x} = x - 1.$

23. $\frac{7x}{6} = \frac{1}{2} - x^2.$

24. $\frac{2x}{5} + \frac{3x+9}{2} = \frac{20}{x} - 1.$

25. $\frac{1}{2}(x+1) - \frac{x}{3}(2x-1) = \frac{2}{3}.$

26. $\frac{x-2}{4} + \frac{12}{x} = \frac{x}{2}.$

27. $2 - \frac{2x+1}{3} = \frac{1}{2x-1}.$

28. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{82}{9}.$

29. $\frac{5y-6}{y-2} + \frac{y-18}{y^2-4} = 6.$

30. $\frac{x}{x-1} + \frac{x-3}{x^2-1} = \frac{5}{4}.$

31. $\frac{3}{x-5} - 5 = \frac{-2x}{x-3}.$

32. $\frac{2y-6}{y+1} = \frac{y+2}{y-1} - \frac{24}{y^2-1}.$

33. $\frac{5x-7}{9} + \frac{14}{2x-3} = x-1.$

34. $\frac{1+x}{2+x} - \frac{3-x}{5-x} = \frac{1}{6}.$

35. $\frac{t-2}{t+2} - \frac{t-1}{t+1} = 6.$

36. $\frac{z-5}{z+3} + \frac{z-8}{z-3} = \frac{3(1-z)}{z^2-9}.$

37. $\frac{3x-2}{x+4} + \frac{x+5}{x-5} = \frac{4x^2+4x-6}{x^2-x-20}.$

38. $\frac{a}{x} + \frac{x}{a} = \frac{33a^2 - x^2}{ax}.$

39. $\frac{m}{x-n} + \frac{n}{x-m} - 2 = 0$

40. $\frac{x-4k+l}{l-k} = \frac{9l+x}{x}.$

Find the value to the nearest hundredth of the roots of examples 41–45.

41. $x^2 + 5x + 1 = 0$.

Solution: By formula

$$x = \frac{-5 \pm \sqrt{25 - 4}}{2}$$

$$= \frac{-5 \pm \sqrt{21}}{2}$$

$$= \frac{\sqrt{21} - 5}{2} \text{ or } \frac{-\sqrt{21} - 5}{2}$$

$$= \frac{4.582 - 5}{2} \text{ or } \frac{-4.582 - 5}{2}$$

$$= \frac{-.418}{2} \text{ or } \frac{-9.582}{2}$$

$$= -.209 \text{ or } -4.791$$

$$= -.21 \text{ or } -4.79.$$

42. $x^2 + 6x + 1 = 0$.

43. $6x^2 + 3x = 5$.

44. $x^2 + 3x + 1 = 0$.

45. $2x^2 = 7x + 2$.

Review

1. Rationalize the denominator of $\dfrac{\sqrt{6} - 4\sqrt{3}}{2\sqrt{6} + 5\sqrt{3}}$ and find its value correct to the nearest hundredth.

2. $\sqrt{46 - 2\sqrt{45}} = ?$

3. (a) $\sqrt{(18.146)^2} = ?$

(b) What is the value of (a) correct to the nearest hundredth? To the nearest tenth?

4. For what values of x are $\dfrac{x}{5} + \dfrac{10}{x}$ and $(9 \div 2\frac{1}{4}) + \frac{3}{6}$ equal to one another?

101. Irrational Equations. — An *irrational equation* is an equation in which the unknown quantity occurs in the radicand of an irrational expression; as, for example,

$$4\sqrt{x-3} = x.$$

Such an equation may be simplified by squaring both members of the equation, to remove the radical signs, after which it may be solved by methods already familiar.

In this connection you should notice that the processes of squaring a quantity and extracting its square root are exactly opposite processes, as are multiplication and division. For that reason when the square root of a quantity is extracted, and the result squared, the quantity obtained must be exactly the same as the one with which we started.

Hence $(\sqrt{x-3})^2 = x - 3$. We therefore have the

Rule. — *To square a quantity under a radical of the second order simply omit the radical sign.*

Obviously a radical sign of the third order may be omitted in like manner when the expression is cubed.

Illustrative example:

Solve and check, $4\sqrt{x-3} = x$.

Solution:

Squaring both members, we have $16(x-3) = x^2$,

or $16x - 48 = x^2$.

Rearranging, we then have $x^2 - 16x + 48 = 0$.

Factoring, $(x-12)(x-4) = 0$.

Solving, $x - 12 = 0$. $x - 4 = 0$.

$x = 12$. $x = 4$.

Check with 12: *Check* with 4:

$4\sqrt{12-3} = 12$. $4\sqrt{4-3} = 4$.

$4\sqrt{9} = 12$. $4\sqrt{1} = 4$.

$12 = 12$. $4 = 4$.

102. Checking Solutions of Irrational Equations. — Checking is of the greatest importance in solving irrational equations, as it frequently happens that numbers obtained from the solution do not check. For instance, if we solve the equation

$$\sqrt{x+6} = 2x - 3.$$

Squaring, $$x + 6 = 4x^2 - 12x + 9,$$

$$4x^2 - 13x + 3 = 0.$$

We obtain $$x = 3 \text{ or } \tfrac{1}{4}.$$

3 will check in the original equation, but $\frac{1}{4}$ will not check. Again if we solve the equation,

$$(x-2) + \sqrt{x^2 - 8} = 0,$$

$$\sqrt{x^2 - 8} = 2 - x.$$

Squaring, $$x^2 - 8 = 4 - 4x + x^2.$$

Hence $$4x = 12,$$

or $$x = 3.$$

This value will not check in the original equation. You should notice that this equation could not be true for $x = 3$, as we should have the sum of two positive quantities set equal to zero.

These values, which do not check, are introduced in certain cases by the process of squaring. They are spoken of as *extraneous results*, and must be rejected. You must not forget that the final test as to whether a quantity is a root of an equation or not, lies in the process of checking. If the quantity satisfies the original equation, it is a root; otherwise it is not a root.

NOTE. You will observe that the extraneous result 3 will check in the equation $(x-2) - \sqrt{x^2 - 8} = 0$. Thus in solving an irrational equation, we may obtain not only its roots, but also the roots of the other equations obtained by changing the signs of the radical in all possible ways.

In some cases it may be necessary to square the equation twice before it becomes completely rational.

Illustrative example:

Solve the equation $\sqrt{2x-5} = 5 - \sqrt{2x}$.

Solution:

Squaring, $2x - 5 = 25 - 10\sqrt{2x} + 2x.$

Transposing and collecting terms,

$$10\sqrt{2x} = 30.$$

Dividing by 10, $\sqrt{2x} = 3.$

Squaring, $2x = 9.$

Dividing by 2, $x = \frac{9}{2}.$

Check:

$$\sqrt{2 \cdot \tfrac{9}{2} - 5} = 5 - \sqrt{2 \cdot \tfrac{9}{2}}.$$

$$\sqrt{9 - 5} = 5 - \sqrt{9}.$$

$$\sqrt{4} = 5 - 3.$$

$$2 = 2.$$

Oral Exercises

Solve:

1. $\sqrt{x} = 2.$
2. $\sqrt{x} = 3.$
3. $\sqrt{x+5} = 5.$
4. $\sqrt{x-2} = 4.$
5. $\sqrt{x-1} = 2.$
6. $\sqrt[3]{x-2} = 3.$
7. $\sqrt[4]{x-1} = 2.$
8. $\sqrt{x} - 3 = 0.$
9. $\sqrt{x} - 2 = 0.$
10. $\sqrt{x} - 5 = 0.$
11. $\sqrt{x} - 1 = 3.$
12. $\sqrt{x} - 2 = 2.$
13. $\sqrt{x} - 4 = 3.$
14. $\sqrt{x-3} - 2 = 0.$
15. $\sqrt{x-1} - 3 = 0.$
16. $\sqrt{x-4} - 2 = 0.$
17. $\sqrt{x-3} - 1 = 0.$
18. $\sqrt{x-5} - 2 = 3.$
19. $\sqrt{x-1} - 3 = 4.$
20. $\sqrt{x-6} - 2 = 5.$
21. $\sqrt{5-x} = 4.$
22. $3\sqrt{x} = 1.$
23. $\sqrt{2x} = \sqrt{5}.$
24. $\sqrt{4x} - 8 = 0.$
25. $2\sqrt{x} - 6 = 0.$

Exercises

Some of the equations which follow result in quadratic equations after being rationalized, while others reduce to equations of the first degree.

Solve and check each of the following equations:

1. $\sqrt{6x+1} = 7.$
2. $3\sqrt{x+1} = \sqrt{6}.$
3. $\sqrt[3]{2x-1} = 5.$
7. $\sqrt{x^2+20} = 6.$
8. $\sqrt{x+3} + 3 = x.$
9. $\sqrt{x+4} + x = 8.$

Hint: Cube each member of the equation.

4. $5\sqrt{10-x} + 2 = 12.$
5. $2\sqrt[3]{3x-9} = -4.$
6. $11 - 2\sqrt{5x-4} = 3.$
10. $\sqrt{x+25} = 5 - x.$
11. $\sqrt{x+2} - \sqrt{16-x} = 0.$
12. $\sqrt{x} + 2 = \sqrt{x+12}.$
13. $\sqrt{x-3} = \sqrt{3x+1} - \sqrt{2x+4}.$
14. $\sqrt{2x-6} = \sqrt{5x-3}.$
15. $3 - 2\sqrt{x} = -\sqrt{x+9}.$
16. $\sqrt{x} + \sqrt{x-12} = \dfrac{24}{\sqrt{x}}.$
17. $\sqrt{x+5} - \sqrt{x-2} = 1.$
18. $\dfrac{3}{\sqrt{x-4}} + \sqrt{x-4} - 4 = 0.$
19. $\sqrt{x+2} - 2\sqrt{x-3} + 1 = 0.$
20. $\sqrt[4]{5x+1} - \sqrt{2x-2} = 0.$
21. $\sqrt[3]{3x+1} + \sqrt{x+7} = 0.$
22. $\sqrt{-2 + \sqrt{7 + \sqrt{3 + \sqrt{x+4}}}} = 1.$
23. $\dfrac{\sqrt{x+8}}{\sqrt{7x+8}} = \dfrac{\sqrt{x+17}}{\sqrt{12x+4}}.$
24. $\dfrac{x-2}{\sqrt{x}} = \dfrac{m-2}{\sqrt{m}}.$
25. $2\sqrt{2x+5} = 5\sqrt{5x-6} - \sqrt{3x+10}.$
26. $\sqrt{\dfrac{3x-4}{x-5}} + \sqrt{\dfrac{x-5}{3x-4}} = \dfrac{5}{2}.$

Review

1. Solve $\dfrac{x^2+1}{x} = \dfrac{a}{c} + \dfrac{c}{a}$.

2. A man invests five twelfths of a certain sum in $4\frac{1}{2}\%$ bonds and the balance in $3\frac{1}{2}\%$ bonds, and has an annual income therefrom of \$117.50. How much is invested in each kind of bond, provided the bonds are considered at par?

3. Find correct to the nearest hundredth the value of

$$\sqrt{\frac{(a+b)^{-\frac{1}{3}}}{(11a+b^2)^{\frac{1}{6}}} \cdot \frac{(a^3-b^3)^{-\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}} \quad \text{if } a = 5 \text{ and } b = 3.$$

4. Simplify: $21\sqrt{\tfrac{2}{3}} - 5\sqrt{\tfrac{4}{5}} + 6\sqrt{4\tfrac{1}{6}} - 10\sqrt{3\tfrac{1}{5}} + \tfrac{4}{3}\sqrt{11\tfrac{1}{4}}$.

5. Will either or both of the values -3 and 4 satisfy the equation $x - \sqrt{(25 - x^2)} = 1$? Prove.

6. Simplify: $\dfrac{x-y}{1-\dfrac{y^2}{x^2}} + \dfrac{x^2-y^2}{\dfrac{y}{x}-1}$.

103. The Quadratic Form. — If the equation $(x^2-1)^2 + 3(x^2-1) - 270 = 0$ be simplified by removal of parentheses, you would find that it is not a quadratic equation, but one of the fourth degree in x. If, however, we substitute some letter such as m for the quantity $x^2 - 1$, the equation becomes $m^2 + 3m - 270 = 0$, and this is a quadratic equation in m. Such an equation is said to be in *the quadratic form*. An equation is in the quadratic form if it may be written $a(\ \ )^2 + b(\ \ ) + c = 0$, where any like quantities whatever occupy the parentheses.

Equations in the quadratic form can be solved by the methods already explained for solving quadratic equations.

We shall now illustrate this principle by the solution of the above example.

Illustrative example I:

Solve $(x^2 - 1)^2 + 3(x^2 - 1) - 270 = 0$.

Solution:

Substitute m for $x^2 - 1$.

Then $m^2 + 3m - 270 = 0$.

Factoring, $(m + 18)(m - 15) = 0$.

Hence $m = 15$ or -18.

But $m = x^2 - 1$.

Hence $x^2 - 1 = 15$, or $x^2 - 1 = -18$.

$\therefore x^2 = 16$, or $x^2 = -17$.

And $x = \pm 4$, or $\pm\sqrt{-17}$.

Illustrative example II:

Solve $x - 1 + 10\sqrt{x - 1} = 39$.

Solution:

Substituting m for $\sqrt{x - 1}$, this may be written

$m^2 + 10m - 39 = 0$.

Then $(m + 13)(m - 3) = 0$.

And $m = -13$ or 3.

$\therefore \sqrt{x - 1} = -13$ or 3.

Squaring, $x - 1 = 169$ or 9.

$x = 170$ or 10.

170 does not check, hence 10 is the only root which satisfies the original equation.

Sometimes an equation not in the quadratic form may be made so by a simple change.

Illustrative example III:

Solve $(4x + 1) - 3\sqrt{4x + 5} = 6$.

Solution: In order to make the quantity within the parentheses like that under the radical, add 4 to each member of the equation. We obtain $(4x + 5) - 3\sqrt{4x + 5} = 10$, and this equation is in the quadratic form.

Substituting as usual,

$$m^2 - 3m - 10 = 0.$$
$$(m - 5)(m + 2) = 0.$$
$$m = 5 \text{ or } -2.$$
$$\therefore \sqrt{4x + 5} = 5 \text{ or } -2.$$
$$4x + 5 = 25 \text{ or } 4.$$
$$4x = 20 \text{ or } -1.$$
$$x = 5 \text{ or } -\tfrac{1}{4}.$$

The only root is 5, since $-\frac{1}{4}$ does not check.

Illustrative example IV:

Solve $x^{\frac{2}{3}} + x^{-\frac{2}{3}} = \frac{17}{4}$.

Solution:

This may be written $$x^{\frac{2}{3}} + \frac{1}{x^{\frac{2}{3}}} = \frac{17}{4}.$$

Clearing of fractions and arranging terms, we have

$$4x^{\frac{4}{3}} - 17x^{\frac{2}{3}} + 4 = 0.$$

Let $m = x^{\frac{2}{3}}$. Then $$4m^2 - 17m + 4 = 0.$$
$$(4m - 1)(m - 4) = 0.$$
$$m = 4 \text{ or } \tfrac{1}{4}.$$
$$x^{\frac{2}{3}} = 4 \text{ or } \tfrac{1}{4}.$$

Then $$x = \pm 8 \text{ or } \pm \tfrac{1}{8}.$$

It is possible to solve examples of this type without making a substitution, as shown in

Illustrative example V:

Solve $4x + 3 - 6\sqrt{4x + 1} = -3$.

Solution: Note that by adding a -2 to the $+3$ in the left-hand member, we would have the same quantity as we find under the radical sign, viz. $4x + 1$.

In order to add this -2, we must also add -2 to the right-hand member to preserve the equality. We do this and we have

$$4x + 1 - 6\sqrt{4x+1} = -5.$$

Transposing, $4x + 1 - 6\sqrt{4x+1} + 5 = 0,$

or $(4x+1) - 6(4x+1)^{\frac{1}{2}} + 5 = 0.$

Factoring, $[(4x+1)^{\frac{1}{2}} - 5][(4x+1)^{\frac{1}{2}} - 1] = 0.$

$(4x+1)^{\frac{1}{2}} - 5 = 0.$	$(4x+1)^{\frac{1}{2}} - 1 = 0.$
$(4x+1)^{\frac{1}{2}} = 5.$	$(4x+1)^{\frac{1}{2}} = 1.$
Squaring, $4x + 1 = 25.$	$4x + 1 = 1.$
$4x = 24.$	$4x = 0.$
$x = 6.$	$x = 0.$

Check with $x = 6$: $4 \cdot 6 + 3 - 6\sqrt{4 \cdot 6 + 1} = -3.$

$27 - 6 \cdot 5 = -3.$

$-3 = -3.$

Check with $x = 0$: $0 + 3 - 6\sqrt{0+1} = -3.$

$3 - 6 = -3.$

$-3 = -3.$

Exercises

Solve the following equations:

1. $x^4 - 3x^2 = 4.$
2. $x^6 - 7x^3 - 8 = 0.$ (Only 2 roots required.)
3. $x + 2\sqrt{x} = 15.$
4. $(x-2)^2 + 3(x-2) - 18 = 0.$
5. $(2x^2 + 7) + 2\sqrt{2x^2 + 7} = 35.$
6. $(4x^2 + 1)^2 - (4x^2 + 1) - 2 = 0.$
7. $x^3 - 2x^{\frac{3}{2}} = 48.$
8. $4x - 3x^{\frac{1}{2}} - 27 = 0.$
9. $x^{-2} + 2x^{-1} = 8.$
10. $\sqrt{x} - \sqrt[4]{x} = 2.$
11. $3x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} - 6 = 0.$
12. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 9 = 0.$
13. $8x^{-3} + 19x^{-\frac{3}{2}} - 27 = 0.$
14. $3y^3 - 11\sqrt{y^3} = 104.$
15. $(x^2 + x) + 3\sqrt{x^2 + x + 6} = 48.$

16. $6y-\sqrt{6y+4}=52$.

17. $7x^{\frac{1}{2}}-3x^{-\frac{1}{2}}+4=0$.

18. $3x^{\frac{1}{3}}=4+4x^{-\frac{1}{3}}$.

19. $\frac{5}{x}+\frac{6}{\sqrt{x}}-11=0$.

20. $\left(x+\frac{1}{x}\right)^2-5\left(x+\frac{1}{x}\right)+\frac{50}{9}=0$.

21. $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-15=0$.

22. $x^{\frac{1}{4}}+3x^{\frac{1}{2}}-14=0$.

Hint: Note arrangement.

23. $x^{\frac{2}{7}}-2x^{\frac{1}{7}}=3$.

24. $(4x+3)^{\frac{1}{2}}=2x$.

25. $(3x^2-2x)^{\frac{1}{3}}=2$.

26. $(x+3)^2+8(x+3)=-12$.

27. $(x-5)^2+5(x-5)=6$.

28. $3x^{\frac{1}{4}}+10x^{\frac{1}{2}}-13=0$.

29. $10x^{\frac{2}{3}}+7x^{\frac{1}{3}}=12$.

30. $x+5+\sqrt{x+5}-12=0$.

31. $x^2-12-\sqrt{x^2-12}=12$.

32. $x^2-5x+4=10\sqrt{x^2-5x-5}$.

33. $\frac{x^2+3}{12x}+\frac{12x}{x^2+3}=-\frac{10}{3}$.

Hint: Let $y=\frac{x^2+3}{12x}$.

34. $\sqrt[3]{(3x^2-2)^{-1}}=2(3x^2-2)^{\frac{1}{3}}$.

Review

1. Solve by the formula, $3x^2-14x=24$.
2. Solve by factoring, $x^2=3bx-2b^2$.
3. Solve $x^4-7x^2+12=0$.
4. Solve $\sqrt{x-12}-\sqrt{x+9}=\sqrt{x-7}$.
5. Solve and check, $x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0$.

Solve:

6. $x\sqrt{x}-2\sqrt{x}=x$.

7. $\frac{12}{\sqrt{x^2+8}}=\sqrt{x^2+8}+x$.

8. $\frac{ax+b}{cx+d}=1$, if $a=c, b=d$.

9. $\frac{x-4}{2}-\frac{x-2}{6}=\frac{1}{x-3}$.

10. $(x+1)(3x-2)-50=0$.

Word Problems

1. The remainder is $x + 2y$, the subtrahend is $-x - 3y$, what is the minuend?

2. The area of a certain rectangle is denoted by $x^3 - y^3$, the width is denoted by $x - y$, what denotes the length?

3. The average in certain examples is obtained by dividing the total of the total column by the total of the number of men. What is the average wage in the following?

Number of Men	Daily Wage	Total
x	$\$y$	
$2x$	$\$a$	
$23x$	$\$b$	
$25x$	$\$c$	
$30x$	$\$d$	

4. If $x = 2$, $y = \$30$, $a = \$6$, $b = \$5$, $c = \$4$, and $d = \$3$ in the above example, what is the average daily wage? The median daily wage? Are there more men receiving nearer the average daily wage or the median daily wage?

5. How do you obtain a fourth root? A sixth root?

6. What is the value of $6.447x$ correct to the nearest tenth?

7. A man is now $(x + 2)$ years old. How many years ago was he 25 years old?

8. Write the condition that d dimes shall equal n nickels.

9. If x represents an even whole number, what represents the next larger odd whole number?

10. If a merchant mixes 10 pounds of coffee worth x cents a pound, 15 pounds worth y cents a pound, and 20 pounds worth z cents a pound, what is the meaning of

$$\frac{10x + 15y + 20z}{45}?$$

11. A salesman sells \$5000 worth of goods. He receives $r\%$ of the sales as his pay. How much does he receive?

12. What is the sum of the square roots of $\sqrt{16 \cdot 9}$? The negative root minus the positive root equals what?

13. The divisor is $x - c$, the quotient is $x - b$, the remainder is $-bc$, what is the dividend?

14. If the average price of imported olive oil in a certain year is $\$x$, and some years later it is $\$3x$, what is the per cent of increase?

15. What is the value of the positive square root of 25 divided by the positive square root of 64, correct to the nearest hundredth?

16. A room is a yards long and b yards wide. If a rug 9 feet by 12 feet be placed in the middle part of this floor, how many square yards of the floor will remain uncovered?

17. If \$500 at interest for 5 days is the same as \$2500 at interest for 1 day, what is the 1-day equivalent of $\$700\,x$ at interest for $(y + z)$ days?

18. If the cost is c dollars, the profit, based on the cost, is x dollars, what is the fractional equivalent of the profit based on the selling price?

19. A man failed in business, owing $\$x$. His resources, or what he owns, is $\$y$. What fractional part of his debts can he pay? If he owes Mr. Jones $\$z$, what will represent the amount that Mr. Jones should receive?

20. If $g\%$ represents gain %, c = cost, $r\%$ = rate of discount, $m.\ p.$ = marked price, state the meaning of the following formula in words: $\dfrac{g\%\text{ of }c + c}{1 - r\%} = m.\ p.$

21. Evaluate $m.\ p.$ in the above formula if

$$g = 40, \quad r = 15, \quad c = \$450.$$

Exercises

Problems leading to quadratic equations.

You should check each result in the problem as stated, and reject those that are meaningless, or which fail to satisfy the conditions of the problem.

1. The perimeter of a rectangle is 40 feet and its area is 96 square feet. Find the length and width.

2. The sum of the reciprocals of two consecutive even whole numbers is $\frac{13}{84}$. Find the numbers.

3. The sum of two numbers is 13 and the sum of their squares is 85. What are the numbers?

4. The product of two consecutive whole numbers is 34 more than the next consecutive whole number. Find them.

5. There are two numbers whose difference is 5, and the difference of their squares is 45. What are they?

6. If three equal squares are placed side by side in a row, the result is a rectangle whose area in square rods is 15 times its perimeter in rods. Find a side of one of the squares.

7. One number is two larger than a second number. The sum of the squares of these numbers is 19 more than the products of the numbers. Find the numbers.

8. The length of a rectangle exceeds its width by 7 feet, and the area of the rectangle is 120 square feet. What are the dimensions?

9. A man wishes to build a bin which will hold 9 tons of coal. The bin is to be 6 feet deep and 4 feet longer than it is wide. Allowing 40 cubic feet for 1 ton of coal, what must be the dimensions of the bin?

10. A piece of tin 9 inches by 12 inches is to be made into an open box with a base of 60 square inches by cutting equal squares from the four corners and then bending up the edges.

What is the size, to the nearest tenth of an inch, of the square cut from each corner?

11. A boy set out 600 tomato plants. The number of plants in each row was 10 less than twice the number of rows. How many plants in each row?

12. The expenses of a camping party for a troop of Boy Scouts were 60 dollars. If there had been three more in the party, it would have cost each boy 1 dollar less. How many were in the party?

13. A man bought two farms for \$4800 each. One farm contained 20 acres more than the other, and cost him \$8 less per acre. How many acres in each farm?

14. A bale of cotton contains 21 cubic feet. Its length is $4\frac{1}{2}$ feet, and its width is $\frac{1}{3}$ of a foot less than its thickness. Find its width and thickness.

15. A motion picture film 100 feet long contains a certain number of individual pictures. If each picture were $\frac{1}{4}$ of an inch shorter, the same film would contain 400 more pictures. How long is each picture?

16. It takes A one day longer to do a piece of work than B requires to do the same work. Both men can do the work together in $2\frac{8}{11}$ days. How many days does each man alone require?

17. A crew can row 8 miles down a stream and back again in 6 hours. If the rate of the stream is 1 mile an hour, find the rate at which the crew can row in still water.

18. A boy was sent with \$2.10 to buy oranges; he found the price 3 cents per dozen higher than he expected, so he bought one half dozen fewer than he intended to buy and received 1 cent in change. What price did he pay for the oranges?

19. A rectangular plot is 15 feet long and 10 feet wide. It is surrounded by a concrete walk of uniform width. The area of the walk is equal to the area of the plot. How wide is the plot?

20. One leg of a right triangle is 1 foot longer than the other leg, and the hypotenuse is 2 feet longer than the shorter leg. Find the sides of the triangle.

21. Two trains run at uniform rates over the same 180 miles of track. One goes 5 miles an hour faster than the other, and requires $\frac{1}{2}$ hour less time to make the run. Find the rate of each train.

22. A rectangular park 80 rods long and 60 rods wide is surrounded by a driveway of uniform width. The area of the driveway is 12 per cent of the area of the park. What is the width of the driveway?

23. If John could run a rod in $\frac{3}{10}$ of a second less time, his rate would be $1\frac{7}{8}$ miles per hour faster. At what rate can he run?

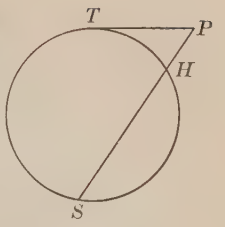
24. A man having sold for \$144 a cow which cost him x dollars, finds that he has made x per cent profit based on the cost. Find x.

25. One leg of a right triangle is 1 foot longer than the shorter leg. The hypotenuse is 9 feet longer than the shorter leg. What are the three sides of the triangle?

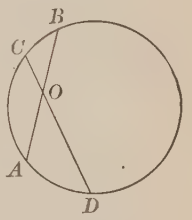
26. What would be the length of the diagonal of the floor of a room 9 feet by 12 feet?

27. A circular pond is surrounded by a path 3 feet wide. The area of the pond is $\frac{6}{5}$ of the area of the path. Find the diameter of the pond to the nearest tenth.

28. In your geometry you learned that from a point P without a circle, a tangent PT and a secant PS cutting the circle at H be drawn, then $\overline{PT}^2 = PH \cdot PS$. If $PT = 5$ and $HS = 4$, find PS.



29. If two chords, AB and CD, intersect at O, $OA \cdot OB = OC \cdot OD$. If $OC = 4$, $OD = 14$, and $AB = 16$. What is the value of OA and OB?



30. One leg of a right triangle is 4 feet more than the other leg. The area of the triangle is 22.5 square feet. Find the hypotenuse.

31. The area of a trapezoid is 30 square feet. One base is 3 feet more than the altitude, and the other base is twice the altitude. Find the bases and the altitude.

32. Find the area of a square if the diagonal is 10 feet more than the side.

33. The area of a rectangle is 2 square feet. The perimeter is 9 feet. Find the sides.

Hint: Remember that the half perimeter is $\frac{9}{2}$.

34. Separate the number 247 into two factors whose difference is 6.

35. If the diagonals of a rectangle are represented by $x^2 + 9x - 6$ and by $3x^2 + x$, what are the values of x and the diagonals?

36. Use the principle shown in example 28, and find the height of a mountain observed at a point on sea level and 40 miles distant from the top of the mountain.

Hint: Radius of earth may be taken as 3960 miles.

37. A tree is broken over by a windstorm. The tree was 90 feet high and the top of the tree is 25 feet from the foot of the tree. What is the height of the standing part of the tree?

38. The length of a rectangle is 2 feet more than the width. If the width be doubled and the length be increased by 2 feet, the area of the new rectangle is 55 square feet more than the original rectangle. What are the dimensions of the original rectangle?

Review

1. Compute the value of $4^{\frac{7}{2}} - 100^{.5} - 81^{\frac{3}{4}} + 36^{1.5}$.
2. Multiply $x^{\frac{3}{4}} - x^{\frac{1}{2}} + 2\,x^{\frac{1}{4}} + 3$ by $x^{\frac{1}{4}} - 2$.
3. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.
4. Simplify $4^{\frac{1}{2}} + 27^{\frac{2}{3}} - 9^{-1} + (-64)^{-\frac{1}{3}} + 15^{0}$.
5. Complete the following tabulated form:

x	x^2	x^3	x^4	x^5	x^0	$x^{\frac{1}{2}}$
2						
3						
4						
5						

CHAPTER X

GENERAL PROPERTIES OF THE QUADRATIC EQUATION

Now that you are able to solve quadratic equations with ease, we shall devote a brief chapter to some considerations which will give you a clearer idea of the relationship between such equations and their roots. The view which you will obtain in this chapter will be of much help to you in solving equations in your later study of mathematics.

104. Character of the Roots. — Under the title "*character*" or "*nature*" we usually include three points with reference to the roots of a quadratic equation; namely, whether the roots are *real* or *imaginary*, and if real, whether they are *rational* or *irrational*, and whether *equal* or *unequal*.

Since the roots of any quadratic equation may be expressed in general form thus:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

NOTE. r_1 is read r sub one, and r_2, r sub two.

You will see that the question whether the roots are real or imaginary depends upon whether the quantity under the radical sign is positive or negative. In fact, the whole question of the character of the roots may be determined by examination of the value of $b^2 - 4ac$ in the given equation.

The expression $b^2 - 4ac$ is called the *discriminant* of the equation.

Rule. — (1) *If the discriminant is a perfect square or 0, the roots are rational; if not, they are irrational.*

(2) *If* $b^2 - 4ac$ *is greater than 0, the roots are real and unequal.*

(3) *If* $b^2 - 4ac$ *is equal to 0, the roots are real and equal.*

(4) *If* $b^2 - 4ac$ *is less than 0, the roots are imaginary.*

NOTE. In applying the above rule notice that if an equation with real coefficients has imaginary roots, they *must* be unequal.

The terms *rational* and *irrational* are not usually applied to imaginary numbers.

It is understood that the equations to which these rules are applied have real and rational coefficients.

Illustrative examples:

Determine the character of the roots of each of the following equations:

1. $2x^2 - 8x = 7$.
2. $3x^2 - 10x + 50 = 0$.
3. $3x - 2 = -\frac{1}{3x}$.
4. $7x^2 - 20x - 3 = 0$.

Solutions:

1. $2x^2 - 8x - 7 = 0$. $b^2 - 4ac = 64 + 56 = 120$.

Therefore the roots are real, unequal, and irrational.

2. $3x^2 - 10x + 50 = 0$. $b^2 - 4ac = 100 - 600 = -500$.

Therefore the roots are imaginary.

3. Clearing of fractions, we have

$9x^2 - 6x + 1 = 0$. $b^2 - 4ac = 36 - 36 = 0$.

Therefore the roots are real, equal, and rational.

4. $7x^2 - 20x - 3 = 0$. $b^2 - 4ac = 400 + 84 = 484$.

As 484 is a perfect square, the roots are real, unequal, and rational.

105. Determining Literal Coefficients. — In the equation

$$2x^2 - mx + 3 = 0,$$

you will see at once that the value of the discriminant $b^2 - 4ac$, and hence the character of the roots, depends on the value of m.

For example, if m is 1, the roots of the above equation are imaginary, and if $m = 10$, the roots are real, unequal, and irrational.

We are sometimes asked to determine the value of such a literal coefficient so as to make the roots of the equation real, equal, and rational.

Illustrative example I:

Determine the value of m in the equation

$$4x^2 + mx + 9 = 0,$$

so as to make the roots of the equation real, equal, and rational.

Solution:

Since the roots are to be real, equal, and rational, $b^2 - 4ac$ must equal 0.

Substituting in $b^2 - 4ac = 0,$

$$m^2 - 144 = 0.$$

Therefore $m = \pm 12.$

Check:

If $m = +12$, the equation becomes $4x^2 + 12x + 9 = 0$, and the roots are $-\frac{3}{2}$ and $-\frac{3}{2}$.

If $m = -12$, the equation becomes $4x^2 - 12x + 9 = 0$, and the roots are $\frac{3}{2}$ and $\frac{3}{2}$.

Illustrative example II:

Determine the value of m which will make the roots of the equation $(m + 1)x^2 - 20x + \frac{m}{6} = 0$, real and equal.

Solution:

Clearing of fractions, we obtain

$$6(m+1)x^2 - 120\,x + m = 0.$$

$$b^2 - 4\,ac = 0.$$

Therefore $14400 - 24\,m(m+1) = 0.$

$$m(m+1) - 600 = 0.$$

$$m^2 + m - 600 = 0.$$

$$m = 24 \text{ or } -25.$$

Exercises

Without solving, determine the character of the roots in each of the following equations:

1. $6\,x^2 + x = 1.$
2. $3\,x^2 - 5\,x + 1 = 0.$
3. $x^2 + 5 = \dfrac{14\,x}{3}.$
4. $3\,x^2 + 7\,x + 5 = 0.$
5. $49\,x^2 + 1 = 14\,x.$
6. $11\,x^2 + 3 = 12\,x.$
7. $9 + 16\,x^2 = -24\,x.$
8. $3\,x^2 - \frac{1}{3}\,x = \frac{8}{3}.$
9. $3\,x(3\,x - 2) + 5 = 0.$
10. $9\,x^2 - 18\,x + 4 = 0.$
11. $x^2 + x + 1 = 0.$
12. $\dfrac{4\,x^2 + 1}{2} = -2\,x.$
13. $4 = \dfrac{2}{x^2} + \dfrac{7}{x}.$
14. $3\,x^2 - 1 = \dfrac{2\,x - 3}{5}.$
15. $x^2 - 6\,x + 9 = 0.$
16. $x^2 - 5\,x + 6 = 0.$
17. $x^2 - 16 = 0.$
18. Check examples 15, 16, and 17 by solving by factoring.

Determine the value of m which will make the roots of each of the following equations real, equal, and rational.

19. $x^2 - 14\,x - m = 0.$
20. $3\,mx^2 + 6\,x + 1 = 0.$
21. $49\,x^2 - 2\,mx + 36 = 0.$
22. $(m-1)x^2 - 6\,x = -1.$
23. $(m+1)x^2 + \dfrac{3\,mx}{4} + 1 = 0.$
24. $(m+1)x^2 + mx - 1 = 0.$
25. $2\,mx^2 + (m+10)x + 9 = 0.$

26. Solve the equation $x^2 - 2x + 2 = 0$, by the formula. What do you observe about both roots? Check by the use of the discriminant.

27. What value of m in $x^2 - 4x + m$ will make the roots

(a) real and unequal?

(b) real and equal?

106. The Incomplete Quadratic. — When we represent any quadratic equation in the general form $ax^2 + bx + c = 0$, we understand that a, b, and c may represent any real numbers whatever. Either b or c or both may be 0, and we shall still have a quadratic equation, though consisting of one or two terms. Such quadratic equations are called *incomplete*.

If $c = 0$, the equation becomes $ax^2 + bx = 0$.

x is now a factor of the equation, and hence $x = 0$ is a root.

If both b and c are 0, the equation becomes $ax^2 = 0$.

x^2 is now a factor, or x is a factor twice. Hence $x = 0$ is a root twice.

If $b = 0$, but c does not equal 0, the equation reduces to the form $ax^2 + c = 0$.

Hence $x^2 = -\frac{c}{a}$ and $x = \pm\sqrt{-\frac{c}{a}}$. That is, the roots are numerically equal, but opposite in sign. Does the above statement imply that these roots are necessarily imaginary? Why?

If $a = 0$, the equation ceases to be a quadratic.

We now state the following important

Principles. — (1) *One root of the general quadratic equation* $ax^2 + bx + c = 0$, *is 0, if, and only if,* $c = 0$.

(2) *Both roots of the general quadratic equation*

$$ax^2 + bx + c = 0$$

are 0, if, and only if, both b *and* c *are* 0.

(3) *The roots of the general quadratic equation*

$$ax^2 + bx + c = 0$$

are numerically equal but opposite in sign, if, and only if, $b = 0$.

107. Formation of Equations. — By the factor theorem, if m is a number which reduces the expression $ax^2 + bx + c$ to 0, when it is substituted for x, then $x - m$ is a factor of $ax^2 + bx + c$. But any number which reduces $ax^2 + bx + c$ to 0 is a root of the equation $ax^2 + bx + c = 0$.

Hence in any equation, $x -$ a root is a factor of the left member set equal to 0.

If, now, we have given two roots, say a and b, we may write the equation, whose roots they are, by multiplying the factors $(x - a)$ and $(x - b)$ and setting their product equal to 0.

Rule. — *To form an equation from given roots,*

(1) *Write* $x -$ *each root as a factor.*

(2) *Multiply these factors together and set their product equal to 0.*

(3) *Clear of fractions if necessary.*

This principle may be extended to any number of roots.

Illustrative example I:

Write the equation whose roots are 3 and -5.

Solution:

The corresponding factors are $x - 3$ and $x + 5$.

The equation is $(x - 3)(x + 5) = 0$, or $x^2 + 2x - 15 = 0$.

Illustrative example II:

Write the equation whose roots are $\frac{1}{2}$, $\frac{2}{3}$, and -1.

Solution:

The corresponding factors are $(x - \frac{1}{2})$, $(x - \frac{2}{3})$, and $(x + 1)$.

The equation is $(x - \frac{1}{2})(x - \frac{2}{3})(x + 1) = 0$,

or $x^3 - \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3} = 0$,

or $6x^3 - x^2 - 5x + 2 = 0$.

108. Roots and Coefficients. — Since different quadratic equations have different roots, and one quadratic equation differs from another only in coefficients, you will see that the roots must be determined by the coefficients. In fact this relation is very definite, as we shall now see.

We are accustomed to write the roots of a quadratic equation as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This expresses both roots together. Let us separate them, denoting one root by r_1 and the other by r_2.

Then
$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Adding, $r_1 + r_2 = -\dfrac{b}{a}$.

Multiplying, $r_1 r_2 = \dfrac{b^2}{4a^2} - \dfrac{b^2 - 4ac}{4a^2} = \dfrac{b^2 - b^2 + 4ac}{4a^2}$

$$= \frac{c}{a}.$$

If we divide the quadratic equation $ax^2 + bx + c = 0$ by a, we obtain $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$.

Hence the

Principle. — *After a quadratic equation in* x *has been divided by the coefficient of* x^2, *the sum of the roots equals the coefficient of* x *with sign changed* $\left(-\dfrac{b}{a}\right)$ *and the product of the roots equals the constant term* $\left(\dfrac{c}{a}\right)$.

The above principle may also be used to write an equation from given roots.

Illustrative example I:

Write the equation whose roots are $-\frac{1}{2}$ and 3.

Solution:

The equation is to have two roots; hence it will be a quadratic equation, and the first term is x^2.

The sum of the roots is $\frac{5}{2}$; hence the coefficient of x is $-\frac{5}{2}$.

The product of the roots is $-\frac{3}{2}$; hence the third term is $-\frac{3}{2}$.

The equation then is $x^2 - \frac{5}{2}x - \frac{3}{2} = 0,$

or $2x^2 - 5x - 3 = 0.$

This principle also enables us to solve many such examples as the following.

In the equation $x^2 + kx + 18 = 0$, one root is twice the other. Determine the value of k.

Solution:

Denote the roots by r and $2r$ respectively.

Then $-3r = k,$ (1)

and $2r^2 = 18.$ (2)

Hence $r^2 = 9,$

and $r = \pm 3.$

Therefore $k = \mp 9.$ By substituting in (1).

The principle here stated is often very useful in checking the solutions of quadratic equations.

For example, determine whether $\frac{1}{2} + \frac{1}{2}\sqrt{17}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{17}$ are roots of the equation $x^2 - x - 4 = 0$.

Solution:

$(\frac{1}{2} + \frac{1}{2}\sqrt{17}) + (\frac{1}{2} - \frac{1}{2}\sqrt{17}) = 1$, which equals the coefficient of x with sign changed.

Also $(\frac{1}{2} + \frac{1}{2}\sqrt{17})(\frac{1}{2} - \frac{1}{2}\sqrt{17}) = \frac{1}{4} - \frac{17}{4} = -\frac{16}{4} = -4,$ which is the constant term.

Hence these are the roots of the equation.

Exercises

Determine, by inspection, the sum and the product of the roots of the equations:

1. $x^2 + 4x + 4 = 0$.

Check by solving the equation, and testing the roots.

2. $x^2 - 8x + 15 = 0$. Check.
3. $x^2 + 9x + 18 = 0$.
4. $3x^2 + 4x - 6 = 0$.
5. $5x^2 - 3x - 2 = 0$.
6. $2x^2 + 5ax - 6a^2 = 0$.
7. $\dfrac{11x - 2x^2}{x + 2} = x - 2$.
8. $3k = 2x - x^2$.
9. $x^2 + 4x + h - k = 0$.
10. $x^2 - x + .5 = 0$.
11. $4x + 5 = 3x^2$.

Using the method of factors, write the equations whose roots are

12. $3, -1$.
13. $-5, -2$.
14. $\frac{1}{2}, \frac{1}{3}$.
15. $\sqrt{2}, -\sqrt{2}$.
16. $3 - \sqrt{2}, 3 + \sqrt{2}$.
17. $a + b, a - b$.
18. $\frac{1}{2} + \frac{1}{2}\sqrt{3}, \frac{1}{2} - \frac{1}{2}\sqrt{3}$.
19. $3 - \sqrt{-1}, 3 + \sqrt{-1}$.
20. $-1+\sqrt{-2}, -1-\sqrt{-2}$.

Using the relation of roots to coefficients write the equations whose roots are

21. $-3, -2$.
22. $5, \frac{2}{3}$.
23. $-2, -\frac{1}{2}$.
24. $3\sqrt{2}, -3\sqrt{2}$.
25. $-4, -3$.
26. $1.5, 2$.
27. $-\frac{1}{2}, -\frac{1}{3}$.
28. $-\frac{3}{4}, -2$.
29. $2a, -7a$.
30. $-\dfrac{n}{m}, \dfrac{m}{n}$.
31. $3 - \sqrt{3}, 3 + \sqrt{3}$.
32. $\frac{2}{3} + \frac{1}{3}\sqrt{2}, \frac{2}{3} - \frac{1}{3}\sqrt{2}$.
33. $2 + i, 2 - i$.
34. $-3+\sqrt{-1}, -3-\sqrt{-1}$.
35. $-\frac{1}{2}+\sqrt{-3}, -\frac{1}{2}-\sqrt{-3}$.

Determine the value of k in each of the following:

36. $x^2 + kx - 14 = 0$ if one root is 2.

37. $x^2 + 4x + k = 0$ if one root is three times the other.

38. $x^2 + kx - 6 = 0$ if one root exceeds the other by 5.

39. $x^2 + kx - 1 = 0$ if the roots differ only in sign.

40. Are $2 + \sqrt{5}$ and $2 - \sqrt{5}$ roots of the equation

$$x^2 + 4x - 1 = 0?$$

41. Are $-3 + i$ and $-3 - i$ roots of the equation

$$x^2 + 6x - 12 = 0?$$

42. Are $2 + \frac{1}{2}\sqrt{-2}$ and $2 - \frac{1}{2}\sqrt{-2}$ roots of the equation

$$2x^2 - 8x + 9 = 0?$$

43. Determine by the principle of section 108, whether .14 and 2 are or are not roots of $x^2 - 2.14x + .28 = 0$.

44. Determine whether $-\frac{1}{4}$ and $\frac{1}{2}$ are or are not roots of the equation $x^2 - \frac{1}{8} = \frac{1}{4}x$.

Review

1. Solve $x^{\frac{1}{2}} - a^{\frac{1}{2}} = \sqrt{x - b}$.

2. If the base of a certain square is increased by 15 feet and its altitude is decreased 2 feet, the area of the resulting rectangle will be twice the area of the square. Find the side of the square.

3. Form the equation whose roots are $-.05$ and $+.7$.

4. Evaluate $3x^2 + 8x - 4$, when $x = .5$. When $x = \frac{3}{4}$. When $x = 2.5$.

5. The value of a brick building when built is \$10,000. If the yearly depreciation or deduction for wear is 2% of this value, find the value at the end of each year for 8 years, and draw a graph to represent the respective values. What is the probable number of years of life of the building?

6. The hypotenuse of a right triangle is 9, the altitude is 6; find the base correct to the nearest hundredth.

Check by constructing the triangle on squared paper.

7. The hypotenuse of a $30°$–$60°$ right triangle is 10.392 inches. Find the two legs of the triangle. Check with squared paper. What is the area of the triangle?

8. What is the value of

$$(10^2)^2?\quad (10^2)^3?\quad (2^3)^2?\quad (2^4)^3?$$

9. What is the value of

$$(2^{\frac{1}{3}})^2?\quad (16^{\frac{5}{8}})^4?\quad (4^{\frac{1}{3}})^4?\quad (6^{\frac{1}{2}})^4?$$

10. Construct a price graph of gasoline when it is retailing at 19 cents a gallon. Let the x-axis denote number of gallons up to 10, and the y-axis denote 19 cents to each square or unit. From your chart find the cost of 7 gallons of gasoline.

11. Construct a similar graph to denote wages at 45 cents an hour. From your graph find the wages for 8 hours. For 3 hours. For $6\frac{1}{2}$ hours.

12. The cost or premium for each \$1000 of insurance to be paid for 20 years in a certain company is given in the following form:

Age of Insured	Cost per Year per \$1000 20-Year Endowment Policy
20	\$48.48
25	49.33
30	50.43
35	51.91
40	54.06

Draw a curve-line graph to represent these figures.

CHAPTER XI

SYSTEMS OF EQUATIONS

Although many problems which arise can be solved by an equation involving one unknown quantity, situations frequently occur where the relations can be expressed more simply by the use of two or more equations in two or more unknown quantities. The solution of such equations is explained in the present chapter.

109. The Indeterminate Equation. — A single equation in two or more unknowns is called *indeterminate* because more than one set of values for the unknowns will satisfy it. For example, the equation $x - 3y = 10$ is satisfied by the values $x = 10$, $y = 0$; $x = 40$, $y = 10$; $x = 13$, $y = 1$; $x = -23$, $y = -11$; etc. In fact, such an equation is satisfied by an indefinite number of sets of values.

If, however, we have two linear equations in the same two unknowns, there can, in general, be found one and only one set of values which will satisfy both equations. Two or more equations considered together, for which a common solution is desired, is called a system of equations.

HISTORICAL NOTE

It is a peculiar fact that the earliest writers on the subject of algebra gave much more of their attention to the solution of indeterminate equations than they did to those equations which we call determinate.

Diophantus, a Greek (about 350 A.D.), concentrated most of his work on the solution of indeterminate equations. He used a separate method for each problem.

The Hindus were perhaps the first to arrive at general methods for the treatment of indeterminate equations.

It was probably due to René Descartes, a Frenchman (1596–1650), that we are able to do so much more with such equations. His development of the graph of an equation leads us to have a far clearer idea of such an equation as $3x + 5y = 36$ than we otherwise could have, for by charting this equation we can almost instantly find corresponding values which will satisfy it.

110. Systems Impossible of Solution. — Although, as stated above, a system of equations can usually be solved, there are two types of systems which cannot be solved, but which deserve our attention in passing.

Consider the system

$$2x + 3y = 8.$$
$$4x + 6y = 20.$$

Plainly these equations make contradictory statements. If $2x + 3y = 8$, then $4x + 6y$ must equal 16. The statement that $4x + 6y = 20$ is therefore inconsistent with the statement made by the first equation. This is called an *inconsistent system.* Such a system has no solution, since there are no values of x and y which can satisfy both equations.

As a second example consider the system

$$x - y = 9.$$
$$3x - 3y = 27.$$

In this system the second equation is related to the first one, in that it may be obtained by multiplying the first by 3. Any system in which one equation depends upon the other, so that it may be directly obtained from the other, is called a *dependent system.* Such a system is indeterminate, since any set of values which will satisfy one equation will also satisfy the other.

111. Simultaneous Systems. — A set of equations in the same number of unknowns in which each equation is consistent with the others and independent of them is called a *simultaneous system*. Such a system can be solved.

$$x - y = 3$$
$$x + 2y = 9$$

is such a system. Its roots are $x = 5$, $y = 2$.

112. Linear Systems. — A system is said to be *linear* when both the equations which compose it are linear or first-degree equations.

113. Elimination. — The solution of a system of equations is effected by eliminating all of the unknowns but one. For this we have two principal methods. The one called *elimination by addition or subtraction* is illustrated by the following example.

Solve $2x + 3y = -4.$
$5x - 2y = 9.$

Solution: $4x + 6y = -8$ By multiplying the first equation by 2.

$15x - 6y = 27$ By multiplying the second equation by 3.

$19x = 19.$ By adding these equations.

Therefore $x = 1.$

By substituting $x = 1$ in either of the given equations we obtain $y = -2$, and these are the roots of the system.

Check: $2 - 6 = -4.$
$5 + 4 = 9.$

In checking the solution of a system of equations you should substitute the values found in both of the given equations.

NOTE. The numbers by which you multiply are obtained by dividing the L. C. M. of the coefficients of the letter which you are eliminating by each of the coefficients of that letter.

Rule. — (1) *If necessary, multiply either or both of the given equations by numbers which will make the coefficients of one of the unknown quantities alike in both equations.*

(2) *If these like coefficients have like signs, subtract one equation from the other; if they have unlike signs, add the equations.*

(3) *The result will be an equation in one unknown. Solve this equation.*

(4) *Substitute the value thus obtained in one of the given equations to obtain the value of the other unknown.*

(5) *Check in the original equations.*

The second method of elimination is known as *elimination by substitution*, and may be illustrated by the following example.

Solve
$$2x + 5y = 8. \qquad (1)$$
$$x - 10y = 9. \qquad (2)$$

We may solve the second equation for x in terms of y, thus,
$$x = 10y + 9. \qquad (3)$$

Then, since a quantity may be substituted for its equal, we substitute $10y + 9$ in place of x in the first equation.

We obtain
$$2(10y + 9) + 5y = 8.$$
$$20y + 18 + 5y = 8.$$
$$25y = -10.$$
$$y = -\tfrac{2}{5}.$$

Substituting $-\frac{2}{5}$ for y in either of the given equations or in (3), we obtain $x = 5$, and these are the roots of the system.

Check:
$$10 - 2 = 8.$$
$$5 + 4 = 9.$$

The following is a statement of the

Rule. — (1) *Solve one of the equations for one of the unknown quantities in terms of the other.*

(2) *Substitute the expression obtained in place of the quantity for which it stands in the other equation.*

(3) *Solve that equation.*

(4) *Substitute the value thus obtained in one of the given equations or in step* 1 *to obtain the value of the other unknown.*

(5) *Check in original equations.*

Oral Exercises

Find the values of x and y if

1. $x + y = 5.$
$x - y = 1.$

2. $x + y = 6.$
$x - y = 4.$

3. $x - y = 6.$
$x + y = 10.$

4. $x + y = 12.$
$x - y = 2.$

5. $x - y = 10.$
$x + y = 20.$

6. $x - y = 5.$
$x + y = 7.$

7. $x - y = 8.$
$y = 6.$

8. $2x = 10.$
$x + y = 8.$

9. $x + y = 12.$
$3x = 12.$

10. $x + y = 15.$
$4x = 16.$

11. $x + 3y = 12.$
$3y = 6.$

12. $2x - y = 15.$
$2x = 40.$

13. $x + 2y = 12.$
$x - 2y = 2.$

14. $y - 3 = 0.$
$2x + y = 7.$

15. Can any of the following systems be solved? If so, which? Explain your answer.

(a) $x - 5y = 10.$
$2x - 10y = 10.$

(b) $2x + 3y = 8.$
$4x - 10y = 2.$

(c) $\frac{x}{2} - \frac{y}{3} = 10.$
$3x - 2y = 60.$

16. Make up two inconsistent systems, two dependent systems, and two simultaneous systems.

Exercises

Solve each of the following systems by addition and subtraction, and check:

1. $x - 3y = 10.$
   $2x + 3y = 5.$

2. $5x + 3y = -14.$
   $2x + y = 6.$

3. $x - 2y = -13.$
   $3x + y = -4.$

4. $y - 6x = 1.$
   $5x + y = 12.$

5. $y = 12x.$
   $2y - 9x = 7\frac{1}{2}.$

6. $2x + 5y = 20.$
   $3x - 2y = -40.$

Solve by substitution, and check:

7. $3x + 10y = 2.$
   $x - 2y = 6.$

8. $2x - y = 1.$
   $x + 4y = 23.$

9. $3y - x = 23.$
   $2x + 3y = -1.$

10. $5x + 3y = 0.$
    $8x - y = 29.$

11. $4x - y = -2.$
    $10x + 2y = 13.$

12. $x + 6y = -1.$
    $3y - 2x = 12.$

Simplify and solve by either method:

13. $3x - y = 10b.$
    $4x + 9y = 3b.$

14. $3x + 4y = 6m.$
    $\frac{x}{m} - 2 = \frac{3y}{4m}.$

15. $2x - 3y = a - b.$
    $3x - 2y = a + b.$

16. $\frac{2x}{5} + \frac{2y}{3} = \frac{6a}{5}.$
    $\frac{x}{2} + \frac{y}{2} = a.$

17. $lx + 2my = n.$
    $2lx - my = -3n.$

18. $ax + by = k.$
    $cx - dy = l.$

19. $\frac{a}{b+y} = \frac{b}{a-x}.$
    $\frac{a}{b+x} = \frac{b}{a-y}.$

20. $2cx + 2y = 6.$
    $y - 3 = \frac{x}{3}.$

21. $(a-b)x - y = 0.$
$x + (a-b)y = 0.$

22. $\dfrac{x+y+1}{x-y-1} = a.$
$\dfrac{x+y+1}{x+y-1} = b.$

23. $a^2x + b^2y = c^2.$
$p^2x + q^2y = r^2.$

24. $5.7\,x + 2\,y = 10\,h.$
$.19\,x + .4\,y = 2\,h.$

25. $.04\,x - .7\,y = 4.6.$
$.7\,x + .05\,y = -3.4.$

26. $\dfrac{1}{x} - \dfrac{2}{y} = ..$
$\dfrac{3}{x} + \dfrac{4}{y} = 1.$

Solution: $\dfrac{2}{x} - \dfrac{4}{y} = 14$ Multiplying the first equation by 2.

$\dfrac{3}{x} + \dfrac{4}{y} = 1$ Second equation.

$\dfrac{5}{x} = 15$ Adding.

$5 = 15\,x.$ Clearing of fractions.

$x = \frac{1}{3}.$

$3 - \dfrac{2}{y} = 7.$ Substituting the value of x in the first equation.

$3\,y - 2 = 7\,y.$

$-4\,y = 2.$

$y = -\frac{1}{2}.$

In like manner, solve all systems in which the unknowns form the denominators as in the above example:

27. $\dfrac{3}{x} - \dfrac{5}{y} = 3.$
$\dfrac{5}{x} + \dfrac{5}{y} = 1.$

28. $\dfrac{a}{x} + \dfrac{b}{y} = 1.$
$\dfrac{c}{x} + \dfrac{d}{y} = 2.$

29. $x = 3y.$
$\frac{x}{3} + y = 34.$

30. $\frac{x}{4} - \frac{y}{7} = 2.$
$\frac{x}{5} - \frac{y}{3} = -3.$

31. $\frac{x}{7} + \frac{y}{5} = \frac{10}{7}.$
$x + \frac{y}{3} = \frac{14}{3}.$

32. $\frac{1}{x} - \frac{1}{y} = r.$
$\frac{1}{x} + \frac{1}{y} = s.$

33. $\frac{4}{x} - \frac{3}{y} = 2.$
$\frac{10}{x} - \frac{9}{y} = 3.$

34. $\frac{x - 5y}{2} = x - 3.$
$x - 2y = 3\frac{1}{5}.$

35. $\frac{5x}{6} + \frac{y}{4} = 7.$
$\frac{2x}{3} - \frac{y}{8} = 3.$

36. $\frac{x}{2} - \frac{y}{3} = 1.$
$\frac{x}{4} - \frac{y}{9} = 1.$

37. $\frac{x}{5} = \frac{y}{4}.$
$3x + \frac{1}{2}y = 17.$

38. $\frac{t+s}{s-3} = \frac{1}{6}.$
$\frac{2t+5}{s+t} = -9.$

39. $\frac{20x + 9}{7} - \frac{y}{7} = 1.$
$5x - 5y = -29.$

40. $\frac{1}{2}(y+1) + (x-5) = -4.$
$\frac{x+1}{3} + y = 3.$

41. $\frac{m+1}{3} = \frac{2-n}{2} - \frac{5}{6}.$
$\frac{3m-1}{2} = \frac{n-3}{3} - \frac{1}{4}.$

42. $\frac{x-y}{2} = \frac{25}{6} - \frac{x+y}{3}.$
$\frac{x+y-9}{2} - \frac{y-x-6}{3} = 0.$

114. Simultaneous Equations Containing Three or More Unknown Quantities. — In solving a system of equations containing three unknowns, elimination is best performed by the method of addition and subtraction. The solution may be effected by the following

Rule. — (1) *Take two of the given equations and eliminate one of the unknown quantities.*

(2) *Then take a different pair of equations and eliminate the same quantity. The result of these two eliminations will be to obtain two equations having the same two unknowns.*

(3) *Solve these equations to obtain the values of the quantities concerned.*

(4) *Substitute these values in one of the original equations to obtain the value of the third quantity.*

(5) *Check in original equations.*

Illustrative example:

Solve
$$x + y - 3z = 8. \quad (1)$$
$$2x - 3y + z = -6. \quad (2)$$
$$3x + 4y - 2z = 20. \quad (3)$$

Solution:
$$6x - 9y + 3z = -18$$ Multiplying equation (2) by 3.
$$x + y - 3z = 8$$ Add equation (1).
$$7x - 8y = -10 \quad (4)$$
$$4x - 6y + 2z = -12$$ Multiplying equation (2) by 2.
$$3x + 4y - 2z = 20$$ Add equation (3).
$$7x - 2y = 8 \quad (5)$$
$$7x - 8y = -10$$ Subtract equation (5) from equation (4).
$$7x - 2y = 8$$
$$-6y = -18$$
$$y = 3.$$

By substituting $y = 3$ in equation (5) we obtain $x = 2$.

By substituting $y = 3$ and $x = 2$ in (1) we obtain $x = -1$.

Hence, $x = 2$, $y = 3$, and $z = -1$ are the roots of the system, and they will be found to check in all three of the given equations.

You should remember that in all cases of simultaneous equations the number of equations in the system must be the same as the number of unknown quantities.

Exercises

Solve and check each of the following systems of equations:

1. $2x + y - 3z = 5.$
$x + 2y + z = 11.$
$3x - y + 2z = 4.$

2. $2A + 3B - 3C = -2.$
$4A - 2B + C = 9.$
$3A + B + 2C = 13.$

3. $2m + n = -1.$
$3n + 4p = 9.$
$3m - p = 0.$

4. $x - 4y + z = 1.$
$2x + y + 4z = 10.$
$5x - 2y + 3z = -3.$

5. $\frac{1}{x} + \frac{1}{6} - \frac{2}{z} = -7.$
$\frac{2}{x} + \frac{3}{y} + \frac{5}{z} = 43.$
$\frac{5}{x} - \frac{2}{y} + \frac{1}{z} = 10.$

6. $x + y + z = 11.$
$3x - 2y = -3.$
$3x + z = 10.$

7. $2x + 3y + z = 14.$
$x - y + z = 4.$
$x + y - z = 0.$

8. $2x + y + z = 7.$
$x - 3y + 2z = 13.$
$3x + 5y - z = -3.$

9. $2x - 3y = 21.$
$5y + 3z = -1.$
$8x - 5z = -16.$

10. $5s + 3t - 2r = 1.$
$3s + 4r = 14.$
$2t - 7s = -40.$

11. $x + y + z + w = 8.$
$2x - y - 3z + 2w = -1.$
$x + 2y - z - 3w = -13.$
$4x + 2y - 3z - 6w = -24.$

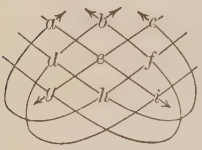
12. $3a + 2b + c - d = -5.$
$2a - 5b + 8c + 2d = 19.$
$7a + 12b - 3c = -5.$
$11a - 7b = 18.$

115. Determinants. — The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ consisting of a square array of quantities with vertical lines down the sides. is called a *determinant*. It is a symbol which represents the expression $ad-bc$. This determinant is said to be of the

second order because it contains two rows and two columns. To expand a determinant of the second order we write the product of the quantities in the *principal diagonal* (ad) with the positive sign, and the product of the quantities in the *secondary diagonal* (bc) with the negative sign.

Similarly the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is of the third order.

Here again aei is the principal diagonal, and the term aei is positive, as are also the products formed by lines parallel to the principal diagonal, indicated by lines in the diagram, namely, gfb and dhc. In like manner the secondary diagonal ceg is negative, as are also the products bdi and ahf. Hence



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \equiv aei + bfg + dhc - ceg - fha - bdi.$$

NOTE. Attention must be paid to the signs of each term in the given determinant.

Exercises

Evaluate each of the following determinants:

1. $\begin{vmatrix} a & c \\ m & n \end{vmatrix}$.
2. $\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix}$.
3. $\begin{vmatrix} 2 & -1 \\ 4 & 6 \end{vmatrix}$.
4. $\begin{vmatrix} -8 & -2 \\ 1 & 5 \end{vmatrix}$.
5. $\begin{vmatrix} -3 & 2 \\ -6 & -7 \end{vmatrix}$.
6. $\begin{vmatrix} x & y & z \\ p & q & r \\ m & n & s \end{vmatrix}$.
7. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 5 \\ 4 & 1 & 2 \end{vmatrix}$.
8. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 0 & 2 \end{vmatrix}$.
9. $\begin{vmatrix} 1 & -2 & 0 \\ 3 & 4 & -5 \\ 0 & 1 & 2 \end{vmatrix}$.

116. Solution of Linear Systems by Determinants. — We may solve the system $\begin{aligned} 2x + 3y &= 13 \\ 4x - y &= 5 \end{aligned}$

thus: $$x = \frac{\begin{vmatrix} 13 & 3 \\ 5 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} 2 & 13 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}}.$$

Notice that the determinants which form the denominators are the same, and in both cases the determinant is formed by the coefficients of the x and y terms in the order in which they stand. The determinant in the numerator in each case is the same as that in the denominator except that the numerical terms replace the coefficients of the letter for which we are solving.

Simplifying the above, we obtain,

$$x = \frac{-13 - 15}{-2 - 12} = \frac{-28}{-14} = +2.$$

$$y = \frac{10 - 52}{-2 - 12} = \frac{-42}{-14} = +3.$$

In a similar manner we solve $\begin{aligned} x + y + z &= 6. \\ 2x + y - z &= 3. \\ x + 2y + 3z &= 13. \end{aligned}$

Thus:

$$x = \frac{\begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & -1 \\ 13 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & -1 \\ 1 & 13 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 2 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}}.$$

Simplifying,

$$x = \frac{18 - 13 + 6 - 13 + 12 - 9}{3 - 1 + 4 - 1 - 6 + 2} = \frac{1}{1} = 1.$$

$$y = \frac{9 - 6 + 26 - 3 + 13 - 36}{1} = \frac{3}{1} = 3.$$

$$z = \frac{13 + 3 + 24 - 6 - 6 - 26}{1} = \frac{2}{1} = 2.$$

Here, as in all other cases of division, the denominator determinant must not be 0.

For solving systems of linear equations by determinants we now state the following

Rule. — (1) *Arrange the given equations in standard form, that is, with the terms containing the unknowns on the left of the = sign, arranged in the same order, and with the numerical terms on the right.*

(2) *Write each unknown as the quotient of two determinants, the denominator being the determinant of the coefficients of the unknown quantities in the order in which they occur, and the numerator in each case being the same determinant as that in the denominator except that the coefficients of the letter for which you are solving are replaced by the numerical terms.*

(3) *Evaluate each of these determinants.*

The above method of solving linear systems gives you just a glimpse of one of the interesting topics to be found in a good course in advanced algebra, and an introduction to a very important method of notation in higher mathematics.

A more complete discussion such as you will get in a later course will show you how the work of evaluating determinants may be much simplified. A discussion of these methods would be out of place here.

Exercises

Solve the following systems of equations by determinants:

1. $x + 2y = 8.$
   $3x + y = 9.$

2. $x + y = 7.$
   $2x + 3y = 18.$

3. $4x + 3y = 7.$
   $5x - 2y = 26.$

4. $x + 4y = 10.$
   $2x + 3y = 5.$

5. $8x - 5y = 1.$
   $\frac{11x - 5y}{22} = \frac{3x + y}{32}.$ *Hint:* First clear of fractions.

6. $.2x + .7y = 5.$
   $.6x + .6y = 6.$

7. $x + y + z = 11.$
   $2x + y - z = 3.$
   $x + 2y + 3z = 25.$

8. $2x + y + z = 5.$
   $2x + 2y + 2z = 6.$
   $x + y + 3z = 11.$

9. $x + y + z = 1.$
   $x + 3y + 2z = 8.$
   $2x + 8y - 3z = 15.$

10. $x + y + z = 6.$
    $4x - y - z = -1.$
    $2x + y - 3z = -6.$

HISTORICAL NOTE

Gottfried Wilhelm Leibnitz (1646–1716), a German, was probably the first to use the notation of determinants. In a letter which he wrote to a friend in 1693 he noted the method of determinants in his attempts to solve certain equations. As this letter was not made public until about 150 years later, it is quite possible that very little was known about them during his life.

Lagrange, a Frenchman (1736–1813), used determinants of the third order in 1773.

Vandermonde (1735–1796), another Frenchman, has been credited with giving one of the first logical treatments of this subject.

Gauss (1777–1855), the famous German mathematician, made more extended use of determinants, and he is credited with being the first to introduce the name *determinants*.

Jacobi (1804–1851), who was born of Jewish parents at Potsdam and died at Berlin, was educated at the university of Berlin. His papers on determinants had much to do with bringing them into general use.

Exercises

Solve each of the following problems by a system of linear equations:

1. 37 coins consisting of nickels and dimes amounted to \$2.30. How many coins of each kind were there?

2. The difference of two numbers is 187 and their quotient is 12. Find the numbers.

3. The width of a rectangle is $\frac{2}{3}$ of its length and its perimeter is 80 yards. Find its dimensions.

4. $\frac{2}{3}$ of one number is 2 less than half of another number, and the sum of the numbers is 67. What are they?

5. Find two numbers whose sum is 114 and whose difference is 58.

6. A man sold sheep and cows, thirteen animals in all, for \$475. For the cows he received \$50 each and for the sheep \$15 each. How many of each kind of animal did he sell?

7. There is a fraction whose value is $\frac{2}{3}$. If 2 be subtracted from the numerator and added to the denominator, its value becomes $\frac{1}{2}$. Find the fraction.

8. 5 pounds of tea and 8 pounds of coffee cost \$4.34, while ten pounds of tea and 3 pounds of coffee cost \$5.04. What is the price of each per pound?

9. Three years ago Mary was three times as old as John. Three years hence she will be twice as old. Find the present age of each.

10. Half of Henry's age added to $\frac{1}{3}$ of Frank's age is 11 years. 6 years from now the sum of their ages will be 40 years. How old is each now?

11. The sum of the numerator and denominator of a certain fraction is 28. If 2 be added to the numerator and subtracted from the denominator, the value of the resulting fraction is $2\frac{1}{2}$. Find the original fraction.

12. The admission to an entertainment was 50¢ for adults and 25¢ for children. The total receipts from 150 tickets were \$62.50. Find the number of adults admitted and the number of children admitted.

13. A man buys two houses for \$15,000. He pays \$650 more for one than he does for the other. Find the value of each house.

14. Two partners, Williams and Jones, wish to divide their profits in the relation of 3 to 4. If they make \$2500, what would each receive?

15. If one of two complementary angles is 10° greater than three times the other, how many degrees in each of the angles. (The sum of two complementary angles is 90°.)

16. The sum of two fractions is $\frac{5}{6}$, and the smaller is $\frac{2}{3}$ of the larger. What are they?

17. The perimeter of a rectangle is 300 feet. The length is 30 feet more than the width. Find the dimensions.

18. The sum of the reciprocals of two numbers is $\frac{10}{21}$ and the difference of their reciprocals is $\frac{4}{21}$. What are the numbers?

19. A man rowed 10 miles up a river in 5 hours and back in $2\frac{1}{2}$ hours. Find the rate of the current and his rate of rowing in still water.

20. A steamer goes downstream 60 miles in 4 hours. The return trip takes two hours longer. What is the rate of the current and the rate of the boat in still water?

21. Flying with the wind a bird was able to make 150 miles an hour, but flying against a wind of only half as great velocity it could make only 30 miles an hour. Find the velocity of the wind and the bird's rate of flying when there was no wind.

22. If the velocity of a cannon report with the wind is 350 yards per second, and 330 yards per second against the wind, what is the velocity of the wind?

23. A boy rows 8 miles with the current in 1 hour and 4 minutes, and returns in $2\frac{2}{7}$ hours. At what rate would he row in still water? What is the rate of the current?

24. A boy started to walk to school at 8 o'clock. At 8.20 his brother started to overtake him, walking two miles an hour faster. He overtook him in 25 minutes. Find the rate at which each boy walked.

25. A and B together can do a piece of work in $3\frac{3}{7}$ days, while if A works 3 days and B 4 days, they will finish the work. Find the time required by each to do the work.

26. John and his father can do a piece of work in $4\frac{1}{2}$ days. After they have worked 3 days the father falls ill, and John finishes the task in 6 days more. How many days does each require to do the entire task?

27. A man had \$8600 to invest. He invested part of it in bonds at par, paying 5%, and the remainder in a mortgage, paying 6%. The income from both investments was \$454. How much was invested at each rate?

28. A part of \$5000 was invested at $4\frac{1}{2}\%$ and a part at $5\frac{1}{2}\%$. The $4\frac{1}{2}\%$ investment yields \$75 more each year than the $5\frac{1}{2}\%$ investment. How much is invested at each rate?

29. The sum of the digits of a certain two-digit number is 10. If 18 be added to the number, the resulting number has the same digits in reverse order. What is the number?

30. If a certain number of two digits be divided by the sum of the digits, the quotient is 2 and the remainder 8. Three times the given number is 2 greater than the number expressed by the same digits in reverse order. What is the number?

31. A certain number of two digits equals 4 times the sum of its digits, while the number made by reversing the order of the digits is 27 greater than the given number. What is the number?

32. Milk testing 3% butter fat is to be mixed with cream testing 28% butter fat. How many quarts of each must be used to make a mixture of 10 gallons which shall test 18% butter fat?

33. The force P necessary to lift a weight W by means of a certain machine is given by the formula $P = a + bW$, where a and b are constants depending on the amount of friction in the machine. If a force of 7 pounds will raise a weight of 20 pounds, and a force of 13 pounds will raise a weight of 50 pounds, what force is necessary to raise a weight of 40 pounds? (First find the values of a and b in that machine.)

34. Find two numbers whose sum is a and whose difference is b.

35. The sum of two numbers is m, and the quotient of the two numbers is n. Find the numbers.

36. A and B have d dollars together. If A gives B n dollars, they then have equal amounts. How much has each?

37. The value of a certain fraction is n. If 1 be added to both numerator and denominator, its value becomes m. What is the fraction?

38. Two books cost together d dollars, and one cost c cents more than the other. Find the cost of each.

39. If A gives B c dollars, A will have twice as much as B. If B gives A d dollars, A will have 6 times as much as B. How much has each?

40. If 6 pounds of butter and 4 dozen eggs cost \$$m$, and 4 pounds of butter and 3 dozen eggs cost \$$n$, what is the price of a pound of butter and of a dozen eggs?

41. The sum of the three angles of any triangle is 180°. In a certain triangle the largest angle is 24° less than the sum of the other two angles, and the sum of the smallest and largest angles is $\frac{11}{7}$ of the third angle. How many degrees in each angle of the triangle?

42. The sum of three numbers is 32. The quotient of two of them is 3, and the third number is $\frac{3}{5}$ of the sum of the other two. Find the numbers.

43. A, B, and C can do a piece of work in $2\frac{2}{3}$ days when they work together. If A and C each work 3 days and B 2 days, they will finish the work; while if A works 2 days and B and C each 1 day, the work will be only $\frac{13}{24}$ done. In how many days can each do the work alone?

Review

1. Factor each of the following: $x^7 - 729\,x$, $4(a+b)^2 - 20(a+b) + 25$, $30\,cd - 25\,d^2 + 16\,a^2 - 9\,c^2$, $x^3 - 4\,x^2 + 5\,x - 2$, $12\,x^{2m} - 43\,x^m y^n + 35\,y^{2n}$.

2. Rationalize the denominator in $\dfrac{\sqrt{10}}{\sqrt{5} - \sqrt{2}}$.

3. Solve and check $\sqrt{x} - \sqrt{x+5} + 1 = 0$.

4. Without solving determine the nature of the roots of $3\,x^2 + 5\,x = 12$.

5. For what value of k will the roots of $3\,x^2 - 2\,x + k = 0$ be equal?

117. Systems of Equations, One Linear and One Quadratic. — An equation in two unknown quantities is quadratic when it contains the square of either of the unknowns or their product. Examples are $x^2 + x + 3y = 10$, $x^2 - 3y^2 = 8$, $xy + 4y^2 = 80$, $xy = -5$.

In solving systems one of whose equations is quadratic and the other linear, you should eliminate by *substitution*, solving the linear equation for one unknown in terms of the other, and substituting the result in the quadratic equation.

The method is sufficiently illustrated by the following

Illustrative example:

Solve $2x + y = 7.$ (1)

$x^2 + xy = 12.$ (2)

Solution: From the first equation, $y = 7 - 2x.$ (3)

Substitute $7 - 2x$ in the place of y in the second equation.

Then $x^2 + x(7 - 2x) = 12.$

$x^2 + 7x - 2x^2 = 12.$

$x^2 - 7x + 12 = 0.$

$(x - 4)(x - 3) = 0.$

Hence $x = 3$ or 4.

By substituting $x = 3$ in (3), we obtain $y = 1$. By substituting $x = 4$, we obtain $y = -1$.

The roots, therefore, are $\left.\begin{matrix} x = 4, \\ y = -1, \end{matrix}\right\}$ or $\left.\begin{matrix} x = 3, \\ y = 1. \end{matrix}\right\}$

You should note that $y = -1$ when $x = 4$, and that $y = 1$ when $x = 3$. The values of x and y must be carefully associated in pairs as indicated above, so that the pair forms a set of roots which will check in the system. Such systems have, in general, two sets of roots.

In solving the linear equation it is best to solve for that letter which will give the simpler expression to substitute.

Exercises

Solve each of the following systems of equations, being careful to associate correctly the roots found:

1. $x - y = 6.$
   $x^2 + y^2 = 26.$
2. $x - y = 1.$
   $x^2 + y^2 = 25.$
3. $x - y = 8.$
   $x^2 + y^2 = 34.$
4. $x + y = 7.$
   $x^2 - y^2 = 21.$
5. $x + y = 9.$
   $x^2 - y^2 = 27.$
6. $x - y = 9.$
   $x^2 + y^2 = 45.$
7. $x - y = 5.$
   $x^2 + y^2 = 53.$
8. $x - y = 1.$
   $xy = 12.$
9. $x - 3y = 1.$
   $x^2 - y^2 = 15.$
10. $x - 4y = 6.$
    $x^2 + 2xy = 36.$
11. $3x - y = 5.$
    $x^2 + y^2 = 25.$
12. $2x - y = 2.$
    $xy + 3y = 10.$
13. $4x = y + 3.$
    $x^2 = 54 - 2y^2.$
14. $2x = y - 1.$
    $x^2 - xy + y^2 = 19.$
15. $2x - y = 6.$
    $4x^2 + 5y^2 = 36.$
16. $x^2 + xy = 12.$
    $2x - 3y = 11.$
17. $xy - 3y^2 = -30.$
    $4x + y = 1.$
18. $\frac{x^2}{2} - \frac{xy}{3} = 12.$
    $x + y = 1.$
19. $\frac{x^2}{3} - \frac{y^2}{4} = -1.$
    $2x + y = 10.$
20. $x + y = 3.$
    $3x^2 = y^2 + 14.$
21. $x^2 + y^2 + y = 10.$
    $x + y + 1 = 0.$
22. $a - b\sqrt{6} = 0.$
    $a^2 + b^2 = 14b.$
23. $3x^2 + y^2 = 40.$
    $x - y\sqrt{3} = 0.$

24. The product of two numbers is $\frac{1}{3}$ and their sum is 4. Find them.

25. The perimeter of a rectangle is 14 feet. The diagonal is 5 feet. Find the dimensions of the rectangle.

Review

1. Form the quadratic equation whose roots are $\frac{1}{2}$ and $\frac{4}{3}$.

2. Find the roots of $.3\,x^2 - .02\,x = .5$ to the nearest hundredth.

3. Solve $\dfrac{3}{x+1} - \dfrac{x+1}{x-1} = \dfrac{x^2}{1-x^2}$.

4. Divide $x^4 + 4\,a^4 - 2\,a^3x$ by $x^2 + 2\,ax + 2\,a^2$ and check the result.

5. Extract the square root of $49\,x^4 - 17\,x^2 - 28\,x^3 + \frac{9}{4} + 6\,x$.

118. Quadratic Systems Solved by Addition and Subtraction. — Systems of equations in which *both* are quadratic may be comparatively difficult of solution. In this book we shall consider only a few simple types. The simplest of these is the type in which one unknown may be completely eliminated by addition or subtraction. This may be made clear by the following

Illustrative example:

Solve
$$y^2 - 3\,x^2 = -3.$$
$$4\,y^2 + x^2 = 40.$$

Solution:
$$12\,y^2 + 3\,x^2 = 120 \quad \text{Multiplying the second equation by 3.}$$
$$y^2 - 3\,x^2 = -1 \quad \text{First equation.}$$
$$13\,y^2 = 117 \quad \text{Adding.}$$
$$y^2 = 9.$$
$$y = \pm 3.$$
$$36 + y^2 = 40.$$

By substituting $y = +3$ in the first equation.

Hence, $y^2 = 4$

and $y = \pm 2.$

By substituting $y = -3$ in the second equation we obtain the same results. Hence the sets of roots are

$$\begin{array}{cccc} x = 2 & x = 2 & x = -2 & x = -2 \\ y = 3 & y = -3 & y = 3 & y = -3 \end{array}$$

In general, systems both of which are quadratic have four sets of roots.

Exercises

Solve the following systems of equations:

1. $x^2 + y^2 = 50.$
   $3x^2 - 2y^2 = -95.$
2. $4x^2 - y^2 = 0.$
   $x^2 + 2y^2 = 81.$
3. $x^2 + xy = 8.$
   $3x^2 - xy = 56.$
4. $2xy + y^2 = 24.$
   $xy - 3y^2 = -2.$
5. $3x^2 + 2y^2 = 11.$
   $5x^2 - y^2 = 1.$
6. $3xy + 2x^2 = 0.$
   $5xy - 3x^2 = -57.$
7. $2x^2 + 3y^2 = 110.$
   $x^2 - 4y^2 = -143.$
8. $x^2 - xy + y^2 = 3.$
   $3x^2 + 2xy - 2y^2 = 14.$
9. $4x^2 + 25y^2 = 100.$
   $x^2 + y^2 = 4.$
10. $3x^2 - 2y^2 = -2.$
    $2x^2 + 3y^2 = 107.$

Review

1. Solve $\dfrac{\sqrt{x-1}}{\sqrt{x-5}} = \dfrac{\sqrt{x-3}}{\sqrt{x-1}}.$
2. Simplify
   $\sqrt{50} + \sqrt[6]{9} - 4\sqrt{\tfrac{1}{2}} + \sqrt[3]{-24} + \sqrt[9]{27} - \sqrt[4]{64}.$
3. Express $\dfrac{a+b}{a^2 - 2b}$ as a fraction whose denominator is $4a^3 - 8ab + 3a^2b - 6b^2$.
4. (a) Solve the formula $S = \pi(R + r)L$ for R in terms of S, r, and L.
   (b) Find the value of R correct to the nearest tenth if $S = 611.45$, $L = 35.65$, and $r = 14.9$.

5. The following table gives the time of sunrise at intervals of two weeks for a certain period of the year:

Date	Nov. 8	Nov. 22	Dec. 6	Dec. 20	Jan. 3	Jan. 17	Jan. 31	Feb. 14
Time of sunrise	6.42	7.02	7.15	7.26	7.30	7.26	7.15	6.59

Represent the table graphically, and from the graph estimate the time of sunrise December 13.

119. Homogeneous Systems. — An equation is *homogeneous* when every term is of the same degree with reference to the letters it contains. Thus $x - 3y = 0$, $x^2 - 6xy + 9y^2 = 0$, $x^3 + 6x^2y - 3xy^2 + 4y^3 = 0$, are homogeneous equations of the first, second, and third degrees respectively. If there is a constant term other than 0, as $x^2 + xy + y^2 = 10$, the equation is not homogeneous, as the constant term is of the 0 degree in x and y. Such equations may, however, be treated as homogeneous equations for purposes of solution.

Systems of equations both of which are quadratic and homogeneous, or homogeneous except for the constant term, may be solved by the following

Rule. — (1) *If either equation is homogeneous, factor it into two linear factors.*

(2) *Set each linear factor equal to 0.*

(3) *Then one of these linear equations with the other given equation forms a system, one linear and one quadratic. Solve this system by the method of substitution as in section 117.*

(4) *In like manner solve the system formed by the other linear equation and the given quadratic equation.*

The resulting four sets of values are the roots of the given system.

(5) *If neither of the given equations is homogeneous, eliminate the constant term by addition or subtraction, thus obtaining*

a homogeneous equation. Factor this equation, and proceed as above.

Cases in which the homogeneous equation cannot be factored are not considered.

Illustrative example I:

Solve $5x^2 + 8xy - 4y^2 = 0.$

$x^2 - xy + 3y^2 = 69.$

Solution: The first equation is homogeneous. Factoring it, we obtain $(5x - 2y)(x + 2y) = 0.$

We now solve $x^2 - xy + 3y^2 = 69$ when $5x - 2y = 0,$

and $x^2 - xy + 3y^2 = 69$ when $x + 2y = 0.$

In the first system, $x = \frac{2y}{5}.$

Substituting $\frac{2y}{5}$ for x, we have

$$\frac{4y^2}{25} - \frac{2y^2}{5} + 3y^2 = 69.$$

$$4y^2 - 10y^2 + 75y^2 = 1725.$$

$$69y^2 = 1725.$$

$$y^2 = 25.$$

$$y = \pm 5.$$

$$x = \frac{2y}{5} = \pm 2.$$

In the second system, $x = -2y.$

Substituting $-2y$ for x, we have

$$4y^2 + 2y^2 + 3y^2 = 69.$$

$$9y^2 = 69.$$

$$y^2 = \tfrac{69}{9}.$$

$$y = \pm \tfrac{1}{3}\sqrt{69}.$$

$$x = -2y = \mp \tfrac{2}{3}\sqrt{69}.$$

Hence the roots are

$$x = 2 \quad x = -2 \quad x = -\tfrac{2}{3}\sqrt{69} \quad x = +\tfrac{2}{3}\sqrt{69}$$
$$y = 5 \quad y = -5 \quad y = +\tfrac{1}{3}\sqrt{69} \quad y = -\tfrac{1}{3}\sqrt{69}$$

Illustrative example II:

Solve $x^2 + 2xy - y^2 = 2.$
$3x^2 + 5y^2 = 32.$

Solution: In this system neither equation is homogeneous.

Multiply first equation by 16, $16x^2 + 32xy - 16y^2 = 32$

Subtract second equation, $3x^2 + 5y^2 = 32$

$$13x^2 + 32xy - 21y^2 = 0$$

Factor resulting homogeneous equation,

$$(x + 3y)(13x - 7y) = 0.$$

Take the linear equations $x + 3y = 0$ and $13x - 7y = 0$ with the simpler of the given equations, $3x^2 + 5y^2 = 32$.

Solving
$$x + 3y = 0,$$
$$3x^2 + 5y^2 = 32,$$

we obtain
$$y = \pm 1;$$
$$x = \mp 3.$$

Solving
$$13x - 7y = 0,$$
$$3x^2 + 5y^2 = 32,$$

we obtain
$$y = \pm \tfrac{13}{31}\sqrt{31},$$
$$x = \pm \tfrac{7}{31}\sqrt{31}.$$

These four sets of values are the roots of the system.

Exercises

Solve each of the following systems of equations:

1. $4x^2 + 8xy + 3y^2 = 0.$
   $x^2 - xy + 2y^2 = 44.$
2. $3x^2 + 2xy = 0.$
   $x^2 + 5xy - 2y^2 = -44.$
3. $3x^2 + 4xy - 20y^2 = 0.$
   $x^2 - xy + 5y^2 = 7.$
4. $x^2 - 9y^2 = 0.$
   $2x^2 + 3xy = 36.$

5. $5x^2 + 7xy + 2y^2 = 0.$
$5x^2 - 2xy + 4y^2 = 35.$

6. $3x^2 + y^2 = 12.$
$5xy - 4x^2 = 11.$

7. $2x^2 + xy + y^2 = 44.$
$x^2 - xy + 2y^2 = 16.$

8. $x^2 - y^2 = 16.$
$3y^2 + 2x^2 - 4xy = 17.$

9. $x^2 + 2y^2 = 18.$
$3x^2 - xy + 2y^2 = 46.$

10. $2x^2 + xy + 5y^2 = 44.$
$x^2 + xy + 2y^2 = 16.$

120. Symmetric Systems. — A *symmetric* equation is one which remains unchanged when x and y are interchanged. Examples are $x + y = 10$, $x^2 + y^2 = 20$, $3xy = 50$, $2x^2 - xy + 2y^2 = 5$, etc. Systems of equations which are symmetric, or symmetric except for signs, may be conveniently solved by a method explained in the following

Illustrative example I:

Solve $x + y = 8.$ (1)
$xy = 12.$ (2)

We shall first obtain the value of $x - y$ thus:

Square (1), $x^2 + 2xy + y^2 = 64$
$4xy = 48$ By multiplying equation (2) by 4.

Subtracting, $x^2 - 2xy + y^2 = 16$

Hence $x - y = \pm 4$ By extracting the square root.

Add (1), $x + y = 8$

$2x = 12$ or $4.$
$x = 6$ or $2.$
$y = 2$ or $6.$

Notice that if the value of either $x + y$ or $x - y$ is given in the problem, we obtain the value of the other, so that we may eliminate by addition. If neither is given, we must find both as in

Illustrative example II:

Solve $x^2 - xy + y^2 = 13.$ (1)

$x^2 + 3xy + y^2 = 61.$ (2)

Subtract (2) from (1), and we obtain

$-4xy = -48.$

Hence $xy = 12.$ (3)

Subtract (3) from (1), and we obtain

$x^2 - 2xy + y^2 = 1.$

Extract root, $x - y = \pm 1.$ (4)

Subtract (3) from (2), and we obtain

$x^2 + 2xy + y^2 = 49.$

Extract square root, $x + y = \pm 7.$ (5)

Add (4) and (5), $2x = \pm 8$ or $\pm 6.$

$x = \pm 4$ or $\pm 3.$

Subtract (4) from (5), $2y = \pm 6$ or $\pm 8.$

$y = \pm 9$ or $\pm 4.$

$\therefore x = 4, -4, 3, -3.$

$y = 9, -9, 5, -4.$

Exercises

Solve each of the following systems of equations:

1. $x + y = 6.$
   $xy = 5.$
2. $x^2 + y^2 = 29.$
   $xy = -10.$
3. $x^2 - xy + y^2 = 13.$
   $x^2 + y^2 = 10.$
4. $x^2 + xy + y^2 = 13.$
   $3x^2 - 4xy + 3y^2 = 123.$
5. $x - y = 6.$
   $x^2 + y^2 = 26.$
6. $xy = \frac{1}{6}.$
   $x^2 + 6xy + y^2 = \frac{49}{36}.$
7. $x^2 + y^2 = 29.$
   $xy - x^2 - y^2 = -19.$
8. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}.$
   $\frac{6}{xy} = -1.$
9. $x^2 + xy + y^2 = 39.$
   $x + y + 2 = 0.$

Hint: Do not clear of fractions.

Review

1. Simplify:

$$\left(1-\frac{6\,a^2+3\,ab+5\,b^2}{8\,a^2+2\,ab-b^2}\right)\cdot\frac{16\,a^2-4\,b^2}{2\,a+3\,b}\div\left(\frac{4\,ab-(5\,a^2+2\,b^2)}{4\,a-b}+a\right).$$

2. State the Factor Theorem.

Factor $x^4 + 2\,x^3 - 8\,x^2 - 10\,x + 15$.

3. Write the equation whose roots are $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

121. Division as an Aid in Solution. — Frequently systems occur in which one equation may be divided by the other, thus greatly simplifying the system.

Illustrative example:

Solve
$$x^2 - y^2 = 9.$$
$$x + y = 3.$$

Solution: By dividing the first equation by the second we obtain
$$x - y = 3.$$

By replacing the first quadratic equation by the linear equation $x - y = 3$, we have the system

$$x - y = 3.$$
$$x + y = 3.$$

The roots of this system are the finite roots of the original system. Such systems are called *equivalent systems.* By solving we obtain $x = 3$, $y = 0$.

Exercises

Solve each of the following systems of equations:

1. $x^2 - y^2 = 16.$
   $x - y = 2.$
2. $x^2 - y^2 = 36.$
   $x + y = 4.$
3. $x^2 - 9\,y^2 = 72.$
   $x + 3\,y = 8.$
4. $x^2 - 16\,y^2 = 24.$
   $x - 4\,y = 4.$

5. $x^2 - xy - 2y^2 = 27.$
$x - 2y = 9.$

6. $x^2 - 4xy + 3y^2 = 18.$
$x - y = 3.$

7. $x^3 + y^3 = 28.$
$x + y = 4.$

8. $x^3 - y^3 = 98.$
$x - y = 2.$

9. $\frac{1}{x^2} - \frac{1}{y^2} = 12.$
$\frac{1}{x} - \frac{1}{y} = 3.$

10. $\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{10}.$
$\frac{1}{x} + \frac{1}{y} = \frac{1}{4}.$

11. $x^3 - y^3 = 56.$
$x^2 + xy + y^2 = 28.$

12. $x^3 + y^3 = 98.$
$x^2 - xy + y^2 = 49.$

13. $\frac{1}{x^3} + \frac{1}{y^3} = 63.$
$\frac{1}{x} + \frac{1}{y} = 3.$

14. $\pi R^2 - \pi r^2 = 64\,\pi.$
$R - r = 4.$

Review

1. Divide $6x^{3n} - 25x^{2n} + 17x^n - 5$ by $2x^n - 5$.

2. Factor:
   (a) $1 - 8x + 16x^2 - y^2.$
   (b) $x^4 + 64a^4b^8.$
   (c) $x^2 + \frac{5}{6}x + \frac{1}{6}.$

3. Simplify: $\left(\frac{x}{y} + 1\right)\left(\frac{x}{y} - 1\right)\left(\frac{y^2}{y^2 - x^2}\right).$

4. A rectangular piece of paper is twice as long as a square piece and 3 inches wider. The area of the rectangular piece is 108 square inches. Find a side of the square piece.

5. Factor the right-hand member of the formula

$$A = \tfrac{1}{2}bh + \tfrac{1}{2}b'h,$$

thus rendering it simpler for computation. Then compute the value of A, when $b = 4\frac{1}{2}$, $b' = 2\frac{3}{4}$, and $h = 6$.

Exercises

Word problems and the use of algebraic symbols.

1. A man makes a trip of x miles. He travels a miles in an automobile, t miles in a train, and b miles in a boat. What is the equation which shows the relation between all of these letters?

2. What is the simple interest on d dollars at 4 per cent for y years?

3. A square room has a side of s feet. What is the area of the square in square yards?

4. A train is running at an average speed of x feet per second. How many miles will it travel in h hours?

5. A train runs at an average speed of m miles per hour. How many miles will it travel in h hours and 30 minutes?

6. If a man can make a certain distance by automobile in d days, what part of the trip can he make in 1 day? In x days?

7. William has d dollars, John owes x dollars. How much better off is William than John? How much less has John than William?

Write in algebraic symbols:

8. The sum of $5x$ and x^2 divided by 5 plus (eight times x diminished by $6x$ divided by 9).

9. The difference of the squares of the sixth power of x and the sixth power of y.

10. The square of the difference of the sixth power of x and the sixth power of y.

11. Let the two consecutive angles of a parallelogram be represented by x and y. From your knowledge of geometry form a correct equation.

12. If A, B, and C have respectively $3x$, $4x - 500$, and $x + 600$ dollars, write in algebraic symbols:

(a) A has \$10 more than B.

(b) If A earns \$30, and B spends \$40, they will then have equal amounts.

13. Form an equation from the fact that 20 subtracted from $2x$ gives the same result as 7 subtracted from x.

14. The cost of a pair of shoes is d dollars. What is the cost of 8 pairs if the price is raised 50 cents a pair?

15. If 38 cubic feet of space will hold 1 ton of a certain kind of coal, how many tons will a bin l feet long, w feet wide, and h yards high hold?

Exercises

Problems leading to systems of equations which may be solved by quadratic methods.

1. The sum of two numbers is 1 and the difference of their squares is 5. What are the numbers?

2. The difference of two numbers is 7 and the sum of their squares is 137. Find them.

3. The sum of two numbers is 8. The square of the first minus twice the square of the second equals 7. Find the numbers.

4. The quotient of two numbers is $\frac{3}{2}$ and their product is 216. What are the numbers?

5. The product of two numbers is 72 and their difference is 14. Find them.

6. If the sum of two numbers be added to the sum of their squares, the result is 68, while if the difference of the same numbers be added to the difference of their squares, the sum is 44. What are the numbers?

7. There is a rectangle whose perimeter is 46 feet and whose area is 112 square feet. What are its dimensions?

8. There is a rectangle whose perimeter expressed in feet is equal to its area in square feet. Its length is 3 feet greater than its width. Find its dimensions.

9. There is a rectangle whose perimeter is 18 inches. If its length is decreased 5 inches and its width is increased 12 inches, its area is doubled. Find its length and width.

10. The area of a rectangle is 20 square inches. If the length be increased by 4 inches and the width by 1 inch, the area will be 48 square inches. Find the sides.

11. The sum of the areas of two squares is 325 square feet. The side of one increased by the side of the other is 25 feet. Find the length of a side of each.

12. A tennis court for 4 players has an area of 2808 square feet. The perimeter of the court is 228 feet. Find the dimensions of the court.

13. The area of two squares is 117 square inches. The sum of their perimeters is 60 inches. Find the length of a side of each square.

14. Construct a rectangle whose perimeter is 22 inches and whose area is 29.25 square inches.

15. The sum of the radii of two circles is 17 inches. The sum of the areas of these two circles equals the area of a circle whose radius is 13 inches. Find the radii.

Hint: Area of a circle $= \pi r^2$.

16. A farmer received \$51 for some potatoes. If he had sold 10 bushels more and had received for each bushel 30 cents more than he did, he would have received \$79.20. How many bushels did he sell and at what price?

17. Find the fraction which is increased by $\frac{11}{60}$ when 1 is added to its numerator and 2 subtracted from its denominator, while it is decreased by $\frac{1}{6}$ when 1 is subtracted from its numerator and 4 added to its denominator.

18. The expenses of a camping party were \$72. If there had been three boys fewer in the party, it would have cost each boy \$2 more than it did. How many boys were in the party, and how much did it cost each one?

19. The annual income from a certain investment is \$64. If the amount invested were \$400 more and the rate 1% less, the annual income would be \$4 less. What is the amount invested and at what rate?

20. The simple interest on \$1000 for a certain number of years at a certain rate is \$250. If the time were three years more and the rate were 1% more, the interest would be \$480. Find the time and the rate.

21. If a number of two digits be multiplied by the sum of its digits, the product is 252. The tens digit exceeds the units digit by 2. Find the number.

22. The front wheel of a carriage makes 100 more revolutions than the rear wheel in going 600 feet, but if the circumference of each is increased by 1 foot, the front wheel will make only 50 more revolutions than the rear wheel. What is the circumference of each wheel?

23. If it requires 5 days more for one man to do a piece of work than it does for another, and both can do the work in 6 days, in how many days can each do the work alone?

24. The square of the diagonal of a certain rectangle is 100. If two such rectangles are placed end to end, the square of the diagonal of the new rectangle is 292. Find the length and width of the original rectangle.

25. The hypotenuse of a right triangle is 20 inches. The sum of the other two sides is 28 inches. Find the lengths of those two sides.

26. The perimeter of a rectangle is 17.6 inches. The area is 14.52 square inches. Find the length and width.

27. The two legs of a right triangle are equal. The sum of the squares of the two legs and the hypotenuse is 256. Find the sides of the triangle.

28. Find the altitude to the side 12 of a triangle whose sides are 12, 14, and 15.

29. Find two numbers whose sum multiplied by their difference equals a, and whose sum divided by their difference equals b.

30. A man found that he could buy 10 more cows than horses for \$1200, and 4 horses cost 50 dollars more than 10 cows. What is the price of each?

31. The product of two numbers which are expressed by the same two digits in reverse order is 2268, and the difference of the digits is 3. What are the numbers?

32. Find the price of oranges per dozen when 6 more oranges for a dollar lowers the price 10 cents per dozen.

33. A rectangular garden plot whose diagonal was 50 yards had plowed around it a strip whose uniform width was $\frac{1}{6}$ that of the original lot. By so doing the area of the plot was increased by $66\frac{2}{3}\%$. What were the dimensions of the original lot?

34. Two boys ran in opposite directions around a rectangular field whose area is 30 acres. They started at one corner and met 28 rods from the opposite corner. One boy's rate is $\frac{2}{3}$ that of the other. Find the length and breadth of the field.

CHAPTER XII

GRAPHS

EQUATIONS and problems may very often be solved conveniently by graphic methods, and such solutions are usually both interesting and instructive. In advanced work in mathematics much more use is made of the graph than formerly. Some simple graphic work is explained in this chapter.

We shall assume that you are already familiar with graphic methods as related to statistics, and especially the method of drawing a graph by locating points with reference to two axes at right angles to each other whose intersection is the origin. Pupils not familiar with this work should consult the authors' "First Course in Algebra," chapter 12.

122. Graphs of Linear Equations. — It is possible to construct an algebraic graph of the changes in any quantity which depends for its value on another quantity. We may solve the equation $2x + 3y = 6$ for y and obtain

$$y = \frac{6 - 2x}{3}.$$

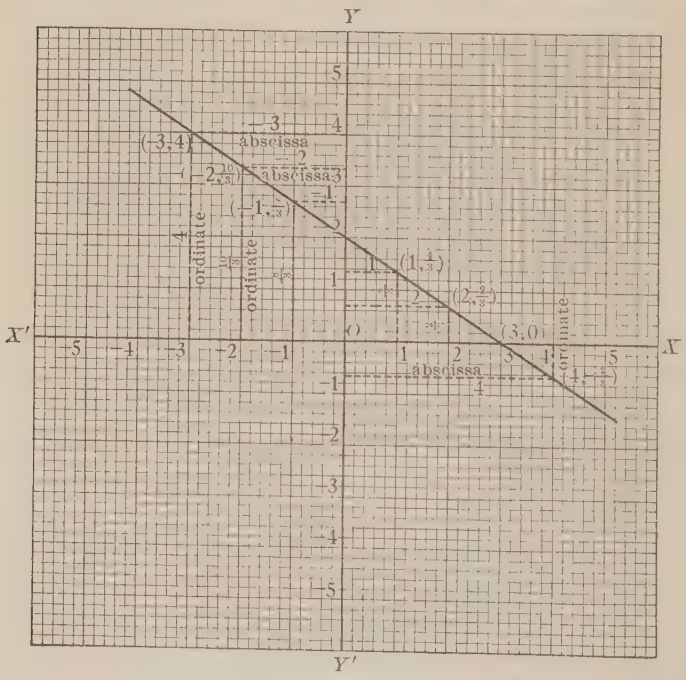
Now if x is given a succession of values, say 0, 1, 2, 3, 4, etc., y will take a corresponding set of values, 2, $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, etc. Hence *the value of y depends on the value of x.* We have therefore the conditions which make a graph possible.

As both x and y may have negative as well as positive values we should draw our axes so that the origin is near the center of the graph sheet. The horizontal axis should be lettered XX' and the vertical axis YY', as indicated. Now

substituting various values for x in the equation $y = \frac{6 - 2x}{3}$, we obtain corresponding values of y, as shown above. We may arrange these in tabular form as follows:

x	0	1	2	3	4	-1	-2	-3
y	2	$\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	4

Values of x are laid off on the horizontal axis, called the *x-axis*, *positive* values to the *right* of the origin and *negative*



Graph of Equation: $2x + 3y = 6$

values to the *left*. In like manner values of y are laid off on the vertical or *y-axis*, *positive* values *above* the origin and

negative values *below*. Each pair of values of x and y enables us to locate one point on the graph.

To locate the point where $x = 0$, $y = 2$, written $(0, 2)$ we locate a point whose distance to the *right* is 0, and whose distance *up* is 2.

The point $(1, \frac{4}{3})$ may be located by measuring off one unit to the *right* and $\frac{4}{3}$ of a unit *up*.

The point $(2, \frac{2}{3})$ is located by measuring off a distance of 2 units to the *right* and $\frac{2}{3}$ of a unit *up*.

In like manner we continue with the other points until we come to the point $(-3, 4)$, which may be located by measuring off 3 units to the *left* and 4 units *up*.

The distances which we have measured off and which are indicated by the dotted lines are called the *coördinates* of the points. The horizontal, or *x-coördinate*, is called the *abscissa*, while the vertical or *y-coördinate* is called the *ordinate*.

If, now, we join these points, we find that our graph is a straight line. It is a general principle that *every equation of the first degree in two unknowns represents a straight line*. For this reason it is called a *linear equation*.

Since we now know that we shall obtain a straight line for every such case, we do not need to locate as many points as in the above example. In fact only two points are necessary to determine a straight line.

If the coördinates of a point are given, the process of locating that point on the chart field is often called *plotting* the point.

Exercises

Draw the graph of each of the following linear equations:

1. $x - y = 1$.
2. $x - 2y = 0$.
3. $3x + y = 3$.
4. $2x + y = 6$.
5. $7x - 2y = 10$.
6. $x - 3y = 8$.
7. $x - y = 0$.
8. $x + y = 0$.
9. $3x + 12y = 0$.

10. $4x - 5y = 20.$ **12.** $3x = 10y.$ **14.** $x = -5.$

11. $9x - y = 6.$ **13.** $x = 4.$ **15.** $y = 3.$

Hint: In the last three examples notice that in each case one of the unknowns must have a constant value, while the other may have any value.

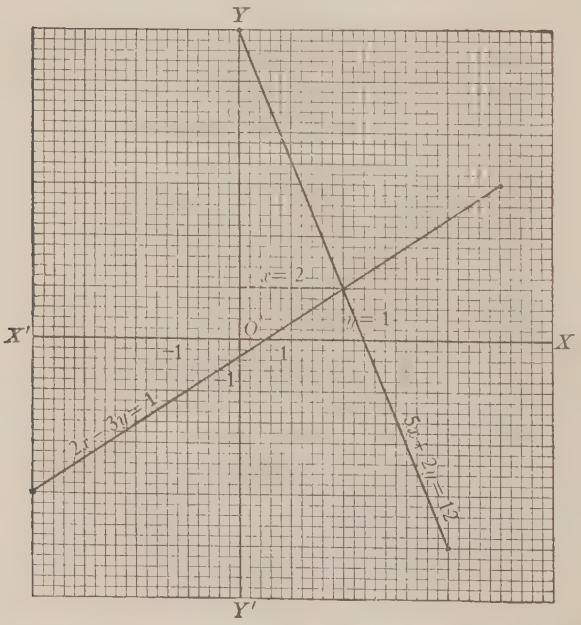
16. Where is the line $y = 0$? $x = 0$?

17. What have you noticed about the graph of a line when the numerical term of its equation is 0?

18. Let it be required to solve by graphic methods the

Illustrative example:

$$2x - 3y = 1.$$
$$5x + 2y = 12.$$



Solution: We construct these graphs on the same axes.

The coördinates of every point on the first graph are the values of x and y which satisfy the first equation. The coördinates of every point on the second graph are the values of x and y which satisfy the second equation. If the lines intersect, the point of intersection is on both lines, hence, its coördinates satisfy both equations and constitute the solution of the system.

In like manner solve graphically each of the following systems:

19. $5x + y = 4.$
$x - 2y = 3.$

20. $3x + 4y = -18.$
$y - x = 6.$

21. $5x - 4y = 8.$
$2x + y = 11.$

22. $4x + 2y = 0.$
$x - 5y = -11.$

23. $5x + 2y = 8.$
$2x - 3y = -12.$

24. $x - y = 3.$
$2x + y = 12.$

25. $x + 2y = -1.$
$5x - 4y = 16.$

26. $3x - 5y = 4.$
$4x + 2y = 1.$

27. $x - 4y = 0.$
$5x + 7y = 0.$

28. $2x + y = 0.$
$4x - 2y = 10.$

29. $2x - 3y = 2.$
$x = 4.$

30. $4x - 5y = 6.$
$y = 2.$

31. Construct graphs of the three equations $x + y = 5$, $3x - y = 3$, and $x - 4y = -10$ on the same set of axes. Which lines cross? What values are common to those lines?

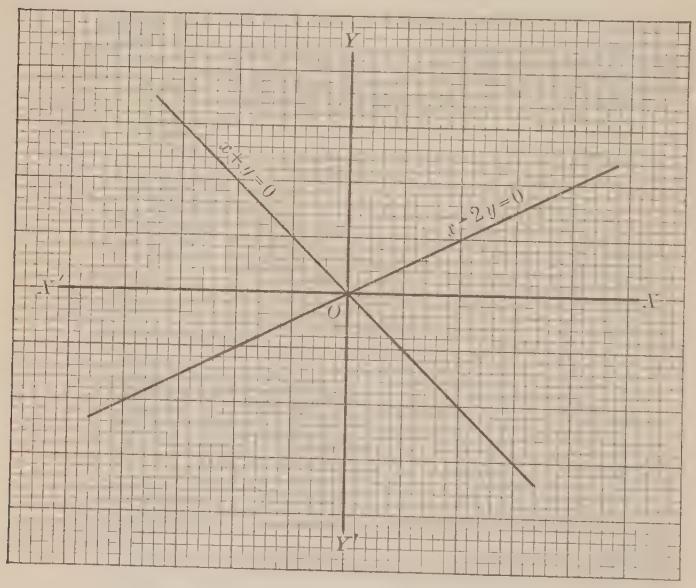
Review

1. Extract the square root of 235.667 to the nearest hundredth.

2. Multiply $\frac{2}{3}x^2 - \frac{2}{3}x + 1$ by $\frac{3}{2}x^2 + \frac{3}{5}x - 1$.

3. A boy starts on his bicycle $3\frac{1}{2}$ hours after his sister, who rides 10 miles an hour, and overtakes her in 5 hours and 50 minutes. How fast did he ride?

123. Graphs of Quadratic Equations. — A quadratic equation in two unknowns, when graphed, always represents one of the following five figures, viz., two straight lines, a circle, an ellipse, a parabola, or a hyperbola. Each of these figures is illustrated in the diagrams which follow.



Illustrative example I:

Graph $x^2 - xy - 2y^2 = 0.$

Factor the equation thus: $(x - 2y)(x + y) = 0.$

Then the graphs of the linear equations $x - 2y = 0$ and $x + y = 0$ constitute the graph of the quadratic equation

$$x^2 - xy - 2y^2 = 0.$$

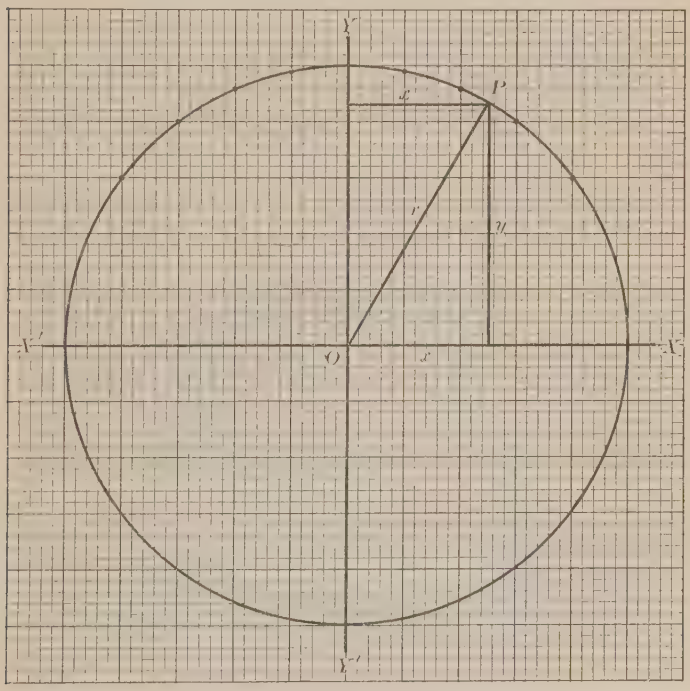
You will observe that when $x = 2$, $y = 1$ in $x - 2y = 0$. Also when $x = 2$, $y = -2$ in $x + y = 0$. You will also note that these values satisfy $x^2 - xy - 2y^2 = 0$. Try other values from the graphs.

Illustrative example II:

Draw graph of $x^2 + y^2 = 25$.

Construct a table of values, substituting in succession for x 1, 2, 3, 4, 5, 8, -1, -2, -3, -4, -5, -6, etc., and obtaining the corresponding values of y.

x	1	2	3	4	5	0	6	-1	-2	-3	-4	-5	-6
y	± 4.9	± 4.6	± 4	± 3	0	± 5	$\pm\sqrt{-11}$	± 4.9	± 4.6	± 4	± 3	0	$\pm\sqrt{-11}$



GRAPH OF $x^2 + y^2 = 25$

You will notice that, since x occurs to the second power only, we obtain the same values for y when a negative value is substituted for x as when the corresponding positive value is substituted.

When we substitute 6 or -6, or any value numerically greater, we obtain imaginary values for y. This shows that the curve does not extend to where x has those values.

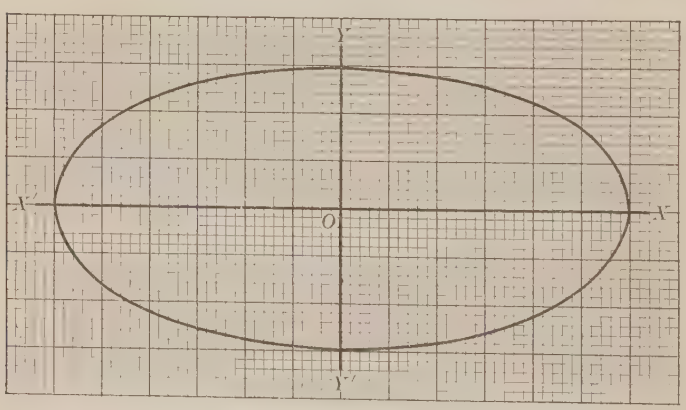
The radius of this circle is 5 units, hence its equation is of the form $x^2 + y^2 = r^2$. Let P be any point on the circle. Draw its ordinate and abscissa and the radius to that point. Is there any reason from geometry why $x^2 + y^2 = r^2$ should represent that circle?

Illustrative example III:

Graph $x^2 + 4y^2 = 36$.

Prepare a table of values as before.

x	0	$\pm$ 2	$\pm$ 3	$\pm$ 4	$\pm$ .5	$\pm$ 6	$\pm$ 7
y	$\pm$ 3	$\pm$ 2.8	$\pm$ 2.6	$\pm$ 2.2	$\pm$ 1.66	0	$\pm \frac{1}{2}\sqrt{-13}$



GRAPH OF $x^2 + 4y^2 = 36$

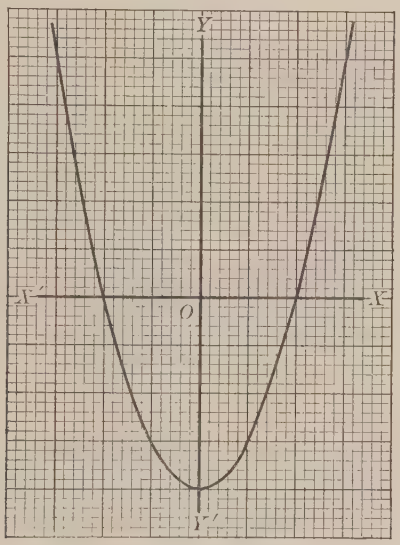
Since we obtain imaginary values for y when $x = \pm 7$, the graph does not extend to those points. In fact it extends only to $x = \pm 6$, and no real value of x gives a value of y numerically greater than ± 3. The figure is called an **ellipse.**

Illustrative example IV:

Graph $y = x^2 - 4$.

The following is the table of values.

x	0	± 1	± 2	± 3	± 4
y	-4	-3	0	5	12



GRAPH OF $y = x^2 - 4$

Since numerically greater values of x give correspondingly greater values of y indefinitely, the figure is an open curve extending without limit. It is called a **parabola**.

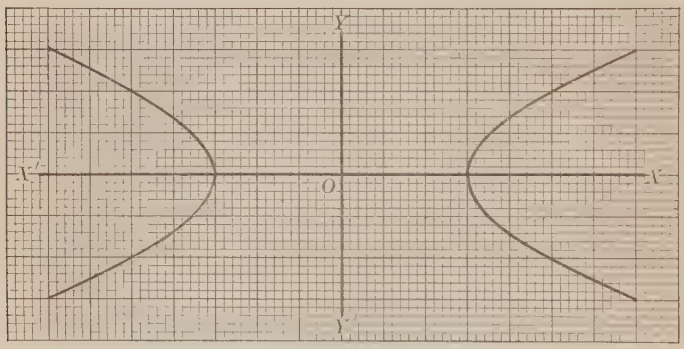
Illustrative example V:

Graph the equation $x^2 - 4y^2 = 9$.

Preparing the usual table of values, we have the following:

x	± 3	± 4	± 5	± 6	± 7
y	0	± 1.3	± 2	± 2.6	± 3.16

Values of x between -3 and $+3$ give us imaginary values of y, showing that the curve does not exist between these points. Beyond these points values of y increase indefinitely as x increases, showing that the curve extends indefinitely in both directions. The result is an open curve of two branches called a **hyperbola.**



GRAPH OF $x^2 - 4y^2 = 9$

124. Determining Character of Graphs of Equations. — It is of great assistance in drawing the graph of an equation to know before we begin to draw it the kind of figure we are to obtain. This can always be determined easily by the following

Rules. — (1) *A quadratic equation in two unknowns represents two straight lines when it can be factored into two linear factors.*

(2) *A quadratic equation in two unknowns represents a circle when the coefficients of* x^2 *and* y^2 *have the same sign and are equal, and there is no* xy *term.*

(3) *A quadratic equation in two unknowns represents an ellipse when the coefficients of* x^2 *and* y^2 *are unequal but of the*

same sign; or, in case they are equal, if there is an xy *term present.*

(4) *A quadratic equation in two unknowns represents a parabola if one of the unknowns occurs squared, and the other to the first power only, and there is no* xy *term.*

(5) *A quadratic equation in two unknowns represents a hyperbola when the coefficients of* x^2 *and* y^2 *have opposite signs, or when either or both of these terms are lacking and an* xy *term is present.*

In applying the above rules the first takes precedence over the others.

Oral Exercises

By use of the above rules determine the character of the graphs of each of the following equations:

1. $x^2 + y^2 = 36.$
2. $x^2 + 6\,y^2 - x + 3\,y = 18.$
3. $x^2 - y^2 + 3\,xy = 18.$
4. $xy = -5.$
5. $x^2 = 4\,y.$
6. $x^2 + xy + y^2 = 25.$
7. $3\,xy - x^2 = 15.$
8. $x^2 + y^2 - 6\,xy + 2\,y = 10.$
9. $3\,x^2 + 5\,y^2 = 28.$
10. $x^2 - xy + 6\,y^2 - 10\,y = 0.$
11. $x^2 - y^2 = 0.$
12. $y^2 = 4\,x + y - 1.$

125. Quadratic Systems Solved by Graphs. — Quadratic systems, like linear systems, may be solved graphically by the following

Rule. — *Draw the graphs of both equations on the same axes. The coördinates of each point of intersection give a set of roots of the system.*

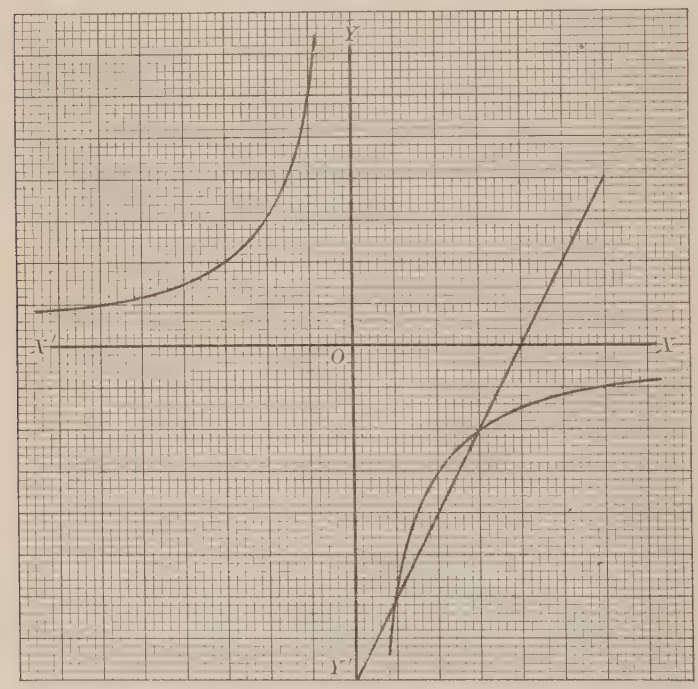
Illustrative example:

Solve graphically the system

$$xy = -6.$$
$$2x - y = 8.$$

Table of values for $xy = -6$ or $y = -\frac{6}{x}$ follows.

x	0	1	2	3	4	6	12	-1	-2	-3	-4	-6	-12	—
y	—	-6	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	6	3	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0



Since division by 0 is impossible, the substitution of 0 for either x or y gives no corresponding value for the other unknown. Hence the graph does not cross either axis. It is a *hyperbola*.

The graph of $2x - y = 8$ is a straight line, since the equation is linear.

You will observe that the graphs intersect at $x = 3$, when $y = -2$, and $x = 1$, when $y = -6$.

$$\therefore \begin{matrix} x = 3 \\ y = -2 \end{matrix} \text{ and } \begin{matrix} x = 1 \\ y = -6 \end{matrix}$$

are the sets of roots of these equations.

Exercises

Draw the graph of each of the following equations:

1. $x^2 = 6y$.
2. $x^2 + 4y^2 = 9$.
3. $x^2 + y^2 = 16$
4. $xy = 4$.
5. $x^2 - y^2 = 16$.
6. $x^2 - xy + 12y^2 = 0$.
7. $y^2 = 4x + 6$.
8. $x^2 - xy + y^2 = 25$.
9. $xy = x - 6$.
10. $x^2 + y^2 - 10x = 0$.
11. $9x^2 + y^2 = 36$.
12. $y^2 - y = 6x$.

Solve graphically each of the following systems:
Check by solving by some other method.

13. $x^2 + y^2 = 25$.
    $2x - y = 5$.
14. $x^2 + 2y^2 = 36$.
    $5x - y = 14$.
15. $y^2 = 4x + 1$.
    $x + 2y = 8$.
16. $xy = 10$.
    $x - 3y = 1$.
17. $x^2 + y^2 = 25$.
    $xy = -12$.
18. $x^2 - y^2 = 16$.
    $x^2 + y^2 = 34$.
19. $x^2 - 9y^2 = 0$.
    $x = y^2 + 2$.
20. $y^2 = x + 4$.
    $x^2 = 5y + 4$.
21. $x^2 + y^2 = 25$.
    $x - y + 1 = 0$.
22. $x + y^2 = 4$.
    $x^2 - y^2 = 16$.

Review

1. Construct the triangle whose vertices are 3, 2; $-6, -5$; 3, -5. Find its area.

2. Simplify $\dfrac{2}{3\sqrt{2}}$. $\qquad \dfrac{2+\sqrt{3}}{2+2\sqrt{3}}$.

3. Combine $\dfrac{3x}{x-1} - \dfrac{2x}{x+1} - \dfrac{3x^2}{1-x^2}$.

4. Factor $x^4 - 47x^2y^2 + y^4$.

5. $\dfrac{L}{4\pi r^2} = \dfrac{75}{360}$. Compute L if $r = 3.5$.

6. Solve for x

$$\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}} = \frac{5}{11}.$$

Which root checks the equation as written? How can you make the other root check?

126. The Function Graph. — When one quantity depends upon another quantity for its value, the first is called a function of the second. For example, the height of the mercury in a thermometer is a function of the temperature, the area of a circle is a function of its radius, and the distance one goes is a function of the time he is going if his rate is constant. (See Section 57.) In our chapter on variation, page 325, you will see another method of expressing this functional relation.

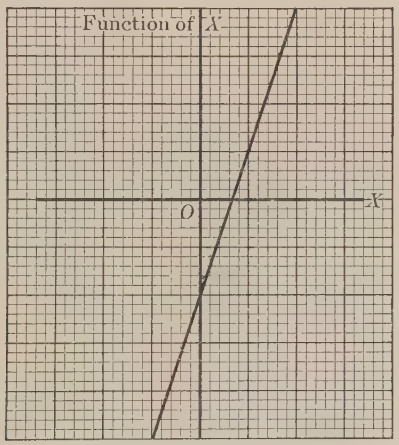
The value of each of the expressions $3x - 2$, x^2, $2x^2 - x + 1$, $\dfrac{5}{x^2}$, is known as soon as the value of x is known. Since their values depend on the value of x, they are all functions of x.

Let us graph the function $3x - 2$. By substituting various values for x we obtain corresponding values for the function.

Thus:

Function of x	-2	1	4	-5	
x	0	1	2	-1	

Since the graph of this function is a straight line, it is called a linear function.



By inspection of the graph answer each of the following questions:

What value of x makes the function 0? 2? -1?

Hint: The function is 0 where it crosses the x-axis. It is 2 where it is two units above it, etc.

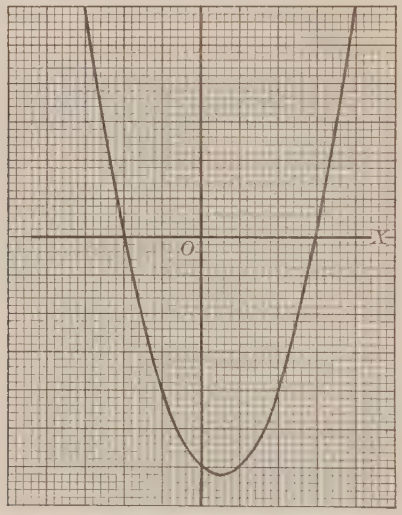
The value of x which makes the function 0 is a root of the equation $3x - 2 = 0$.

You will notice that the graph of the function $3x - 2$ is the same as that of the linear equation $y = 3x - 2$, since we may consider that y is the name of the function.

In a similar manner we may graph the quadratic function

$$x^2 - x - 6.$$

Function of x	-6	-6	-4	0	6	-4	0	6
x	0	1	2	3	4	-1	-2	-3



GRAPH OF THE QUADRATIC FUNCTION
x^2-x-6

The graph of a quadratic function of x is always a parabola. If we call the function y, we obtain the equation $y = x^2 - x - 6$, whose graph is the same as that of the given function.

Consult the graph to answer these questions.

What value of x makes the function have a minimum value? (That is, what value of x corresponds to the lowest point on the graph?)

What values make it 0? What values make it 2? -2?

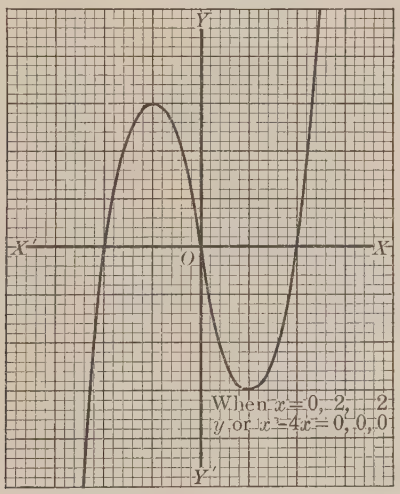
The values of x which make the function $x^2 - x - 6$ equal to 0 are the roots of the equation $x^2 - x - 6 = 0$. By the graph they are 3 and -2.

Graph the function $x^3 - 4x$.

Solution: Form a table of values as follows:

$x =$	0	1	2	3	-1	-2	-3
$x^3 - 4x =$	0	-3	0	15	3	0	-15

Now construct a chart from the above table as shown in the figure.



From the graph determine the values for x when

$$x^3 - 4x = 0.$$

In other words, find out where the curve crosses the x-axis. These will be the roots of the equation $x^3 - 4x = 0$.

You will observe that approximately at $x = -1$ and $x^3 - 4x = +3$, there is a turning point of the curve. In

other words, if we were to let $y = x^3 - 4x$, then $x = -1$ and $y = +3$ would be approximately one turning point.

This point is called a *maximum point*. This does not mean that this is the highest point on the curve, but simply that it is the highest point in that vicinity. That is to say, the graph slopes downward on both sides of this point. In like manner there is a *minimum point* at approximately $x = 1$, $y = -3$. In advanced algebra you will be taught a method of finding these points more exactly.

Rule. — (1) *To solve an equation in one unknown quantity graphically, make the right member 0, and draw the graph of the left member.*

(2) *The points where the graph crosses the x-axis give the values of* **x** *which make the function 0, and hence the roots of the equation.*

Exercises

Solve graphically each of the following equations:

1. $x^2 + x = 0$.
2. $4 - x^2 = 0$.
3. $2x - 5 = 0$.
4. $3x + 8 = 0$.
5. $\frac{x}{2} - 5 = 0$.
6. $x^2 - 9 = 0$.
7. $x^2 + x - 6 = 0$.
8. $x^2 - 2x - 8 = 0$.
9. $x^2 - 4x + 3 = 0$.
10. $x^2 + 3x - 10 = 0$.
11. $2x^2 + 5x - 3 = 0$.
12. $3x^2 + x - 4 = 0$.

In the following the graph is tangent to the x-axis. What fact concerning the roots does that show?

13. $x^2 = 0$.
14. $x^2 - 2x + 1 = 0$.
15. $4x^2 + 4x + 1 = 0$.

In the following, estimate the roots to the nearest tenth:

Hint: See Authors' First Course, p. 203, top line.

16. $x^2 + x - 3 = 0$.

17. $2x^2 - 2x - 1 = 0$.

18. $3x^2 - 2x - 2 = 0$.

19. $x^2 - x - 5 = 0$.

20. $4x^2 = 6$.

21. $3x^2 + 2x - 3 = 0$.

Show from the graph that the following have imaginary roots:

22. $x^2 + 3x + 5 = 0$.

23. $2x^2 + x + 1 = 0$.

24. $3x^2 - x + 2 = 0$.

Construct a graph of each of the following:

25. $x^2 + 3x - 10 = 0$.

26. $x^3 - 3x^2 - 10x + 24 = 0$.

27. $x^3 - 5x + 3 = 0$.

28. $x^3 - 2x - 4 = 0$.

29. Determine the roots of the four above equations, in so far as you are able from the graphs. What do you discover that is peculiar in the graph of example 28? What does this lead you to believe concerning two of the roots of this equation?

30. The length of a rectangle is $x + 1$ and the width is x. Draw a graph to represent the area as x changes from 0 to 10.

31. The edge of a cube is represented by x. Construct a graph which shall show the relation of the volume of that cube as compared with the length of an edge as x changes from 0 to 4.

32. When will the graph of a quadratic equation show us but one real root?

33. When will the graphic representation of a quadratic equation not show us the roots of that equation?

34. Draw a graph of the function $x^2 - 4x + 3$ and see if you can tell for what value of x we find the least (or minimum) possible value.

35. Using the values in order from -3 to $+3$, form a set of values for $y = 2x^3 - 3x^2 - 12x + 12$, and then draw a graph. What value of x gives the largest (or maximum) possible value for the function?

36. What value of x will make the function x^2 have the least possible value?

37. A maker of cardboard boxes wishes to make the largest possible open box out of cardboard 6 inches by 4 inches by cutting out small squares at the corner and turning up the edges, thus forming the box. We wish to find the side of the small square.

Let x equal the side of the square to be cut.

Hint: The length of the box will be $6 - 2x$.

The width of the box will be $4 - 2x$.

The depth of the box will be x.

Hence the volume $V = x(6 - 2x)(4 - 2x)$

$$= 24x - 20x^2 + 4x^3,$$

or $$\frac{V}{4} = 6x - 5x^2 + x^3.$$

Using suggested values of $0, \frac{1}{2}, 1, \frac{3}{2}, 2$, for x, form a table of values for the function and from these draw a graph. Note the points where the graph changes direction and tell what you think should be the value for x in order that he may construct the box.

38. Work a similar problem where the cardboard is 8 inches by 12 inches.

HISTORICAL NOTE

Felix Klein, a famous German professor, was one of the foremost men in the development of the very important study of functions.

Most of the work involving functions is too advanced for this stage of your mathematical knowledge, and for that reason only a very little idea of the subject can be presented at this time.

The subject, however, is of great use in the study of many of the laws of electricity, light, sound, falling bodies, etc. If you should be fortunate enough to pursue the work in mathematics in college some time, you will learn much more about functions, maxima and minima, etc.

Review

1. Solve algebraically $2x^2 - 3x - 1 = 0$, obtaining the roots to the nearest tenth.

2. Solve $\sqrt{x-1} + \sqrt{2x+5} = 8$.

3. If John can spade a garden in 6 hours, and he and James together can do the work in 4 hours, how many hours will James require alone?

4. Simplify $(2 + \sqrt{-1})^3 - 3(2 + \sqrt{-1})^2 - 5(\sqrt{-1})^6$.

5. A rectangular park 56 rods long and 16 rods wide was surrounded by a street of uniform width containing 4 acres. What was the width of the street?

6. Graph

$$x^2 + y^2 = 4.$$
$$x + y = 3.$$

Do these graphs intersect?

Solve algebraically.

What do you conclude from these graphs and your algebraic solution?

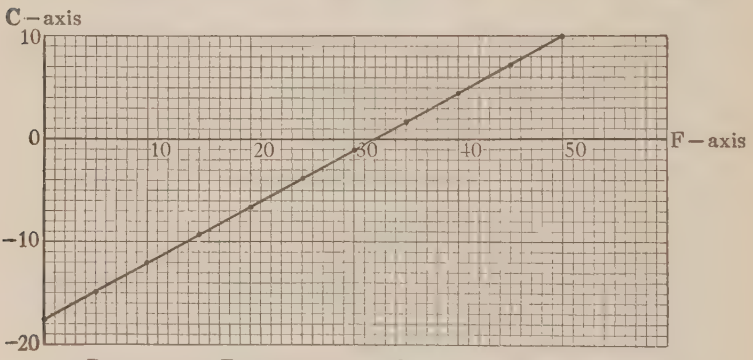
7. A part of \$2000 is invested at 5% and the remainder at 6%. The yearly income from both investments is \$112. How much is invested at each rate?

127. The Formula Graph. — A formula is simply a literal equation in which the letters are usually the initial letters of the words for which they stand. If the formula contains only two letters, it may be graphed exactly as though these letters were x and y.

We shall illustrate this by drawing the graph of the formula $C = \frac{5}{9}(F - 32)$. Since this is a first-degree equation in C and F, the graph is a straight line.

The table of values and the graph follow :

F	0	5	10	15	20	25	30	35	40	45	50
C	$-17\frac{7}{9}$	-15	$-12\frac{2}{9}$	$-9\frac{4}{9}$	$-6\frac{2}{3}$	$-3\frac{8}{9}$	$-1\frac{1}{9}$	$1\frac{2}{3}$	$4\frac{4}{9}$	$7\frac{2}{9}$	10



Relation of Fahrenheit and Centigrade Temperature

The graph shows the relation between temperatures as recorded by a Centigrade thermometer (C) and those recorded by a Fahrenheit thermometer (F).

By means of the graph we can answer such questions as the following :

What temperature Centigrade corresponds to 7 degrees Fahrenheit?

Solution: Find the mark on the F-axis which corresponds to 7 degrees.

Follow the vertical line downward until it intersects the graph. Then follow the horizontal line from the point of intersection until it meets the C-axis. Here we read -14 degrees, which is the desired result. Hence, 7 degrees Fahrenheit equals approximately -14 degrees Centigrade.

In like manner change to Centigrade scale the following Fahrenheit readings: 4 degrees, 13 degrees, 22 degrees, 42 degrees, -1 degree.

Also change the following Centigrade readings to the Fahrenheit scale; -12 degrees, -8 degrees, 5 degrees, 2 degrees.

By extending the graph, find the corresponding Fahrenheit temperatures of the following Centigrade temperatures; $-40°$, room temperature $20°$, freezing temperature of mercury $-39.5°$, boiling temperature of water $100°$, boiling temperature of oxygen $-182.5°$, absolute zero $-273°$, boiling temperature of hydrogen $-252.5°$.

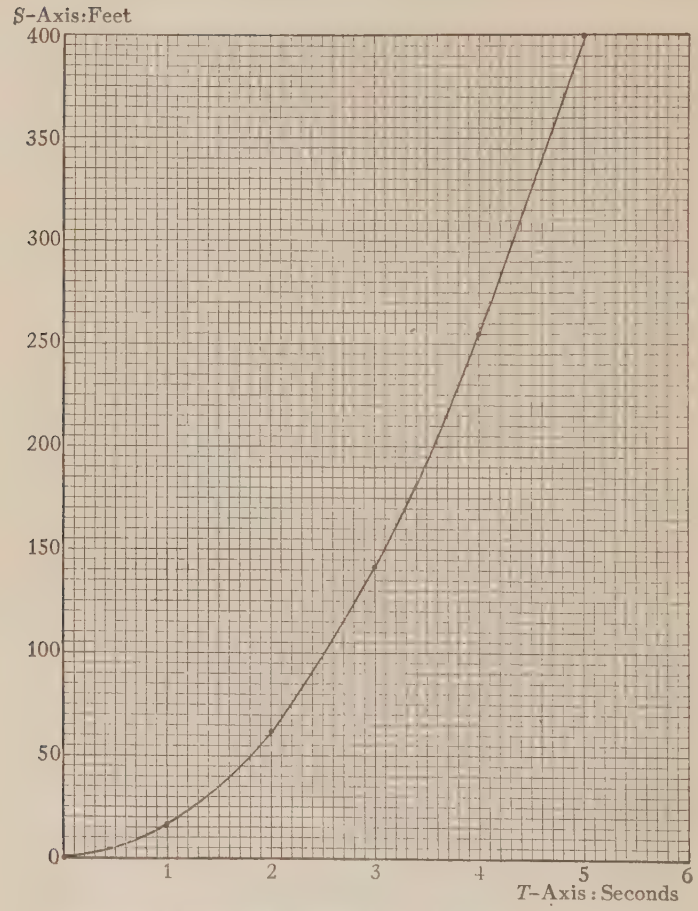
Illustrative example:

We shall draw the graph of the equation $S = 16\,t^2$ which is a formula showing the relation between the distance a body has fallen and the time it has been falling. The rules stated in section 124 show that the graph of this equation is a parabola. Since, however, time which is represented on the horizontal axis cannot be negative, we shall obtain only half of the usual figure.

It may help you to get a better idea of the graph on page 312 if you turn the book upside down.

The table of values and graph follows:

t	0	1	2	3	4	5
S	0	16	64	144	256	400



Relation of Distance a Body Falls to Time It Is Falling

By studying the graph answer the following questions:
How far will a body fall in $2\frac{1}{2}$ seconds? In $4\frac{1}{2}$ seconds?
How long will it take a body to fall 40 feet? 150 feet? 300 feet?

Exercises

Draw graphs to represent each of the following formulas, and discuss them as directed by your teacher.

1. $C = \frac{22}{7} D$, where D = diameter of a circle and C = circumference.

2. $A = \frac{22}{7} r^2$, where r = radius and A = area of a circle.

3. $V = 32\,t$, where V = velocity of a falling body and t = time in seconds.

4. $w = \$.50\,h$, where w = wages and h = hours.

5. $r = np$, where r = receipts, $n = 200$, and p = price per article.

6. $c = 40\,d$, where c = cost and d = number of dozen eggs.

7. $A = s^2$, where A = area and s = side of a square.

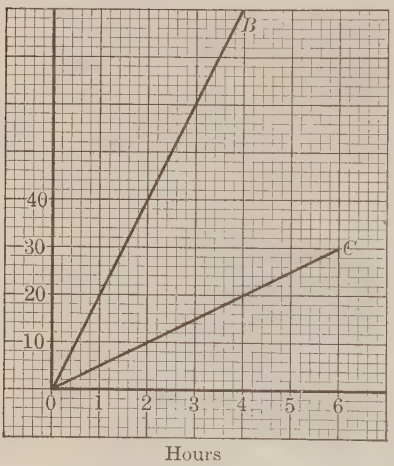
8. $V = e^3$, where V = volume and e = edge of a cube.

9. $V = 200 - 32\,t$, where V = velocity of a body thrown upward with an initial velocity of 200 feet per second, at the end of t seconds.

10. $V = 4(i - 8)^2$, where V = volume of a cardboard box made from a piece of cardboard i inches square, and having 4-inch squares cut out of each corner before folding to form the box.

128. The Problem Graph. — Problems of many kinds, especially those involving the idea of rate, can often be solved conveniently by a graph.

Mark off units of time on the horizontal axis, — in this case hours. On the vertical axis mark off units of distance.



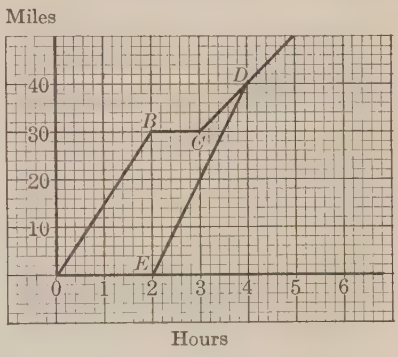
In this case let one large square represent 10 miles. Now suppose we want to represent a rate of 20 miles per hour. In 0 hours at this rate one will go 0 miles, in 1 hour 20 miles, in 2 hours 40 miles, etc. Joining these points we have the straight line OB. In like manner a rate of 5 miles an hour is represented by the line OC. This line does not represent the path of the traveler, but the slant of the line represents the rate. The steeper the line the greater the rate represented.

Illustrative example I:

John leaves a certain point on his bicycle at the rate of 15 miles an hour. Two hours later he has an accident which delays him 1 hour, after which he proceeds at 10 miles an hour. Two hours after John started, James started to overtake him, riding at the rate of 20 miles an hour. How long and how far must he ride?

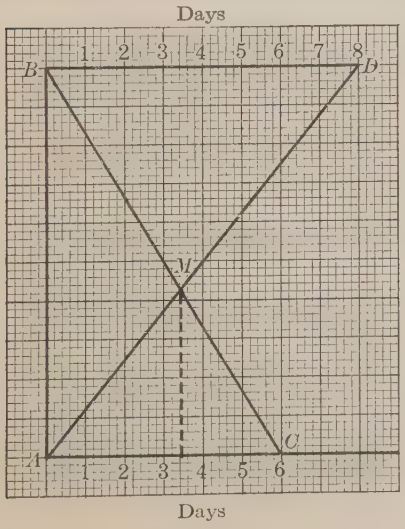
Solution: Draw OB to represent the rate of John and continue it until it reaches the ordinate corresponding to 2 hours. He was then delayed an hour, hence time went on, but distance did not change. The line BC becomes therefore horizontal at this point. Then CD is drawn in a direction to indicate 10 miles an hour, and continued indefinitely.

The fact that James started two hours later than John is indicated by starting the line ED two units to the right from the origin, and the direction of the line ED represents a rate of 20 miles an hour. The intersection D shows that John's rate and James's rate have caused them to reach a distance of 40 miles simultaneously. Hence James has overtaken John. From the axes we read a distance of 40 miles and a time of 2 hours.



Illustrative example II:

A can do a piece of work in 8 days and B in 6 days. In how many days can they do the work if they work together?



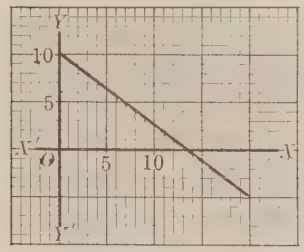
Solution: A convenient distance AB is chosen on the vertical axis to represent the magnitude of the entire task. The line AD is drawn to represent a rate of work which accomplishes the task in 8 days, and BC is drawn to represent a rate which accomplishes it in 6 days. Directly beneath the point of intersection M we read from the

graph that both together can do the work in slightly more than 3.4 days. The algebraic solution shows that the exact result is $3\frac{3}{7}$ days. When solving by graphs, you should be able to read the answer correctly to the nearest tenth.

Illustrative example III:

It is entirely possible to solve some of the problems of the mixtures type by the aid of graphs as shown in the following problem.

Construct a graph to show the different ways that x pounds of coffee worth 30 cents a pound and y pounds of coffee worth 40 cents a pound may be mixed so as to cost \$4.00.



Hint:

$$30x + 40y = 400$$

or

$$y = -\tfrac{3}{4}x + 10.$$

Then form a table of values for x and y and construct the graph of the equation as in the adjoining figure.

State some questions that you may answer by means of the above graph.

Exercises

Solve each of the following problems by a graph:

1. A and B start from the same point and travel in the same direction, one at 20 miles an hour and the other at 24 miles an hour. In how many hours will they be 28 miles apart?

2. Two men, A and B, start at the same time on a journey from M to N, forty miles away. A travels 20 miles an hour, and B 16 miles an hour. When A reaches his destination, he immediately returns at the same rate. How far from N will he meet B?

3. A man travels for two hours at 15 miles an hour, then for one hour at 10 miles an hour, and then for three hours at 30 miles an hour. What uniform rate would have enabled him to make the same distance in the same time?

4. Henry starts to walk to town at 3 miles an hour. After two hours he rests one half hour, and then walks on at 2 miles an hour. Stephen starts 3 hours after Henry and walks 4 miles an hour. After how many hours will he overtake Henry? If the distance to town is 20 miles, how much sooner will Stephen arrive than Henry?

5. A starts from P toward Q at 8 A.M., at the rate of 10 miles an hour. At 9 A.M. B starts from Q toward P at 15 miles an hour. If the distance from P to Q is 60 miles, how far from Q will they meet?

6. One boy can spade a garden in 6 hours, another in 4 hours. How long will it take them to spade the garden if they work together?

7. B works 4 days on a piece of work which he can do in 9 days. Then he is joined by C, who could do the whole work in 8 days. If C works until the task is finished, how many days does he work?

8. Mr. Brown can plow a field in 4 days. Mr. Jones and he together can plow the field in $2\frac{1}{2}$ days. How long will it take Mr. Jones to plow the field alone?

9. Two pipes can fill a tank, one in two hours and 40 minutes, the other in 3 hours and 40 minutes. How long will it take to fill the tank if both pipes are open?

10. Two boys run around a quarter mile track, one making the circuit in $1\frac{1}{4}$ minutes and the other in $1\frac{1}{2}$ minutes. If they start together, how many times around will each have run before they are together again?

11. Coffee worth 25 cents a pound is mixed with coffee worth 30 cents a pound. The total mixture is worth \$4. Construct a graph to show these relations.

Form some questions that may be answered from this chart.

12. Construct a graph of the equation $3x + y = 12$.

Find three sets of values from your graph which will satisfy the equation, in one of which x shall equal 0.

13. Construct a graph of the equation $y = 12 - 3x$. How does it compare with that of example 12?

14. Similar problems containing mixtures of different percentages of butter fat can be graphed by the above method.

A mixture is to be formed which shall contain 24% butter fat from cream containing 20% butter fat and cream containing 30% butter fat. Find the varying amounts of each which may be used.

Hint: Let x represent the number of pints of the 20% cream.

Let y represent the number of pints of the 30% cream.

Then from the problem, $x + y$ represents the number of pints of the 24% cream,

or $$.20x + .30y = .24(x + y).$$

Whence $$.06y = .04x$$

or $$y = \tfrac{2}{3}x.$$

Construct a graph from this equation and give some answers to the problem.

15. A 30% salt solution is to be mixed with pure water so that it shall produce a 20% solution. Construct a graph from the equation formed from the above information and find some varying amounts in pounds that will answer the question.

16. Two American League baseball teams in the playing of a certain game made a total of 16 hits during the game. Form an equation, construct the graph of it, and find some of the possibilities of the number made by each team.

17. It costs \$20 to set up the type and get ready for the printing of a certain booklet. After that it costs 20 cents for each copy. If we let y represent the total cost in dollars and x the number of booklets to be printed, we have the equation $y = .20\,x + 20$.

Construct a graph of this equation and from it determine the cost of 1000 copies. Of 1400 copies. Of 2400 copies. How many copies could be printed for \$40? For \$80? For \$200? Explain the variation in numbers as the cost is increased.

18. If it costs \$5 to prepare the type for the printing of a circular and then 6 mills additional for each copy, construct a graph from the above and find varying costs for varying number of hundreds of copies.

19. A graph may be constructed in a similar manner which shall show the point at which the initial cost has all been covered and the net profit begins. Such a chart is called a *net profit graph.*

If it costs \$300 to prepare special machinery to manufacture a certain article and then the cost of each article is 50 cents, and the article is to be sold for \$1, construct graphs to show where the initial cost has been covered and the net profit begins.

Hint: Construct a graph of the equation $y = .50\,x + 300$. Then construct a graph of another equation $y = mx$, where y represents the total selling price, x represents the number of articles, and m in this example is 1 dollar. Find the point where the two graphs intersect.

20. It costs \$400 to prepare the machinery to manufacture a certain article which is to sell for \$1.20. It costs 30 cents per article thereafter to manufacture it. Draw two graphs and find out how many must be sold before a net profit is shown.

21. If the net profit to the owner of a manufacturing plant is \$3 per day per each workman employed before we take out the overhead expenses, and the average daily overhead expenses is \$8, we can find the net profit for the owner by the equation

$$p = 3n - 8,$$

where p represents the daily profits to the owner,
n represents the number of workmen.

Construct a graph which shall show these relations. What is the profit when 20 men are employed? How many men must be employed to give a profit of \$70?

CHAPTER XIII

RATIO, PROPORTION, AND VARIATION

THE proportional relation of numbers forms the basis of many problems in arithmetic, algebra, and geometry. In this chapter we shall become familiar with such relations, and extend the idea to include the treatment of Variation. These methods of work are very useful in solving problems in physics, chemistry, and other sciences.

129. Ratio. — The *ratio* of two numbers is the quotient obtained by dividing the first by the second. Thus the ratio of 8 to 4 is $\frac{8}{4}$, or 2, while the ratio of 4 to 8 is $\frac{4}{8}$, or $\frac{1}{2}$. The ratio of a to b is often indicated by the symbol $a : b$, which is read "a is to b." Since a ratio really implies division, it may, however, be written just as well in the form $\frac{a}{b}$.

Evidently two numbers unlike in kind cannot have a ratio. There can be no ratio between 2 bushels and 5 yards. We may, however, have a ratio between two different units of the same kind. For example, the ratio of one peck to one quart is 8 : 1.

The first number, or numerator, is called the *antecedent* of the ratio, while the second number, or denominator, is called the *consequent*. Although the antecedent and consequent may be *concrete* numbers, as indicated above, the quotient or ratio is always an *abstract* number.

Principle. — *The value of a ratio is not changed by multiplying or dividing both antecedent and consequent by the same number (except 0).*

Thus the ratio of 10 to 25 is $\frac{10}{25}$, or $\frac{2}{5}$. $\frac{2}{4} = \frac{4}{8}$ or $\frac{1}{2}$.

130. Proportion. — A *proportion* is a statement of the equality of two ratios, as $\frac{a}{b} = \frac{c}{d}$. This may also be written $a : b = c : d$ or $a : b :: c : d$. In any case it may be read "*a is to b as c is to d.*"

In a proportion the first and last terms are called *extremes*, while the second and third terms are called *means*.

It is evident from the above, that a proportion is in reality a fractional equation. Its lowest common denominator is bd. If the equation is cleared of fractions by being multiplied by this common denominator, the result is $ad = bc$. Thus we have proved the

Principle. — *In any proportion the product of the means is equal to the product of the extremes.*

When the two means of a proportion are the same quantity, that quantity is called a *mean proportional* between the other two quantities.

Thus, in the proportion $a : b = b : c$, b is a mean proportional between a and c. In this case also c is called a *third proportional* to a and b. Since in any proportion the product of the means is equal to the product of the extremes, we have here $b^2 = ac$, or $b = \pm\sqrt{ac}$. Hence, the

Principle. — *The mean proportional between two quantities is the square root of their product.*

When a proportion is made up of four *different* quantities, the fourth quantity is called a *fourth proportional* to the other three taken in order. For example, in the proportion $2 : 6 = 5 : 15$, 15 is a fourth proportional to 2, 6, and 5.

Show that a fourth proportional to 2, 6, and 5 is not the same as a fourth proportional to 6, 2, and 5.

A proportion is solved by means of the principles stated above.

Illustrative example I:

Solve the proportion $(x - 1) : (2x + 3) = 2 : 5$.

Solution:
$$5(x - 1) = 2(2x + 3).$$
$$5x - 5 = 4x + 6.$$
$$5x - 4x = 6 + 5.$$
$$x = 11.$$

Illustrative example II:

Divide \$72 between X and Y so that the ratio of X's money to Y's is as 4 is to 5.

Solution: Let z = X's money, and $72 - z$ = Y's money.

Then $$\frac{z}{72 - z} = \frac{4}{5}, \text{ etc.}$$

Exercises

1. Simplify each of the following ratios. 5 to 15, 8 yards to 20 yards, 30 bushels to 25 bushels, 5 feet to 8 inches, 2 rods to 11 feet, 12 ounces to $1\frac{1}{2}$ pounds (avoirdupois).

2. Solve the proportion $(2x - 3) : (x + 1) = 4 : 7$.

3. Solve the proportion $\dfrac{2x + 1}{3} = \dfrac{10x - 5}{6}$.

4. Find a fourth proportional to 4, 6, and 10.

Hint: $4 : 6 = 10 : x$.

5. Find a fourth proportional to $a + b$, $a^2 - b^2$, and ab.

6. Find a third proportional to 8 and 20.

7. Find a third proportional to $(a + 1)^2$ and $a^2 - 1$.

8. Find a mean proportional between 6 and 24, between $4a^2x$ and $100xb^2$, between $(a + b)^2$ and $4(a - b)^2$.

9. Find two numbers in the ratio of 3 to 5 such that when 10 is added to each of them they shall be as 4 is to 5.

Hint: Represent the numbers by $3x$ and $5x$.

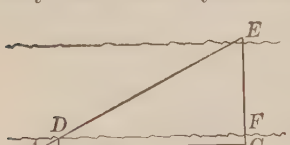
10. Divide 180 into 3 parts which shall have the ratio $2:3:5$.

11. Find two numbers such that if 3 be subtracted from each of them they will be in the ratio $2:1$, while if 5 be added to each of them they will be in the ratio $4:3$.

12. Mr. Jones's will provided that his estate should be divided among his wife, son, and daughter in the ratio $7:5:3$. The son's share was found to be \$6000 more than the daughter's. What was the value of the estate, and how much did each receive?

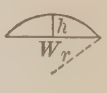
13. The lengths of the sides of a triangle are as 3 to 4 to 5. The total lengths of the sides is 60 feet. Find the length of each side.

14. The distance across a river or an impassible barrier may be found by the use of proportion as follows. The distance AB is to AC as BD is to CE. If $AB = 10$ inches, $BD = 5$ inches, and AC measures 800 feet, what is the width CE? Note that ABD and ACE are right triangles.



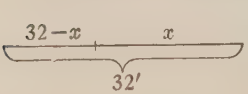
15. If the width of a circular arc is W, the radius is r, and the rise or height is h, as denoted in the adjoining figure, then the half width will be a mean proportional between the height and the diameter − the height, or

$$\frac{h}{\frac{1}{2}W} = \frac{\frac{1}{2}W}{2r - h}.$$

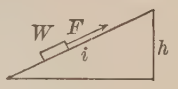


Find r when $W = 40$ feet, and $h = 6$ feet.

16. Find the lengths of the segments of a line 32 feet long, if x, one part, is a mean proportional between the whole line and the other part.



17. A weight of 1200 pounds is to be pushed up an inclined plane. How much force (F) applied parallel to the plane is required to move it up the plane if $l = 40$ feet, $h = 8$ feet, and $\frac{F}{W} = \frac{h}{l}$?

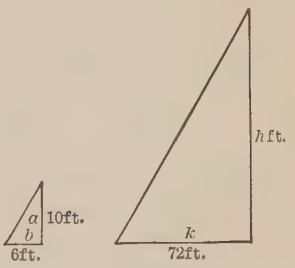


NOTE. No account is to be taken of friction.

18. Find the height of a flagstaff which throws a shadow of 72 feet, when a vertical rod 10 feet long casts a shadow of 6 feet.

Hint: From geometry we know that

$$\frac{b}{k} = \frac{a}{h} \text{ or } \frac{6}{72} = \frac{10}{h}.$$



19. Using a similar method find the height of your flagpole or of some high building.

20. Just as the sun is setting over a distant mountain, which is known to be 5000 feet high, you find that the sun casts a shadow of 30 feet from a vertical pole 10 feet long. How far are you from the center of the mountain?

Variation

131. Variable Quantities. — A *variable* is a quantity whose value is constantly changing. Examples are one's age, the distance one has traveled when on a moving train, the temperature, the population of a city, the price of a commodity which changes from day to day, etc. A quantity whose value does not change is called a *constant*.

132. Related Variables. — Variables are frequently so related that a change in one produces a corresponding change in the other. For example, if one is on a moving train which

travels at a constant rate, the distance he has traveled depends on the time he has been traveling. Then, as we explained in section 126, the distance is a *function* of the time. The same truth may be expressed by saying that the *distance varies as the time.*

Again, the area of a rectangle which has a constant width depends upon its length, and hence is a function of the length. We may express this also by saying that if the width of a rectangle is constant the *area varies as the length.*

133. Variation, as applied to quantity, is a study of the laws which show the way in which two variables are related. The symbol for variation is $\propto$, and the statement $x \propto y$ is read "x varies as y."

There are several ways in which variables may be related to form a variation, and these will be explained in the sections which follow.

134. Direct Variation indicates that the ratio of the two variables is constant. That is, y varies as x when $\frac{x}{y} = k$. Take, for example, the illustration of the moving train in section 132. If the constant rate of the train is 40 miles an hour, then $\frac{d}{t} = 40$.

A direct variation may also be expressed as a proportion: thus, $x : y = k : 1$; or as an equation, $x = ky$.

When one quantity varies directly as another, the ratio of any two corresponding values of these variables will have the same value as the ratio of any other two corresponding values, because in each case the ratio equals the constant k. It follows that any problem involving only direct variation may also be solved by proportion.

Illustrative example:

x varies directly as y, and equals 5 when y equals 8. Find x when y equals 24.

The pair of corresponding values mentioned in the example have the ratio $\frac{5}{8}$. Therefore, the ratio of the unknown number to 24 is also $\frac{5}{8}$. Hence, $\frac{x}{24} = \frac{5}{8}$, and $x = 15$.

The example may also be solved as follows:

Since x varies as y, $x = ky$. Hence, $5 = 8k$ and $k = \frac{5}{8}$.

The constant k in such examples is spoken of as "the constant of variation."

Now in the formula $x = ky$, substitute for k $\frac{5}{8}$, and for y 24, and we obtain $x = \frac{5}{8} \cdot 24 = 15$.

We now state the following

Principle. — *If the first variable varies as the second, then the first equals a constant times the second.*

135. Inverse Variation. — One quantity varies *inversely* as another when, as one increases, the other decreases in the same ratio. In such cases their *product* is constant. For example, if the length and width of a rectangle change so that the area of the rectangle is always 40 square feet, then the length varies inversely as the width. Also, if one takes several trips from New York to Philadelphia under varying conditions of time and rate, then the time varies inversely as the rate since their product, the distance, is constant.

When "x varies inversely as y" we may also say that "x is inversely proportional to y." This means that x is equal to the constant k multiplied by the reciprocal of y, or that $x = \frac{k}{y}$. Also, if x and y, and x' and y' are corresponding values of the variables, we may write the proportion $x : x' = y' : y$.

We now state the following

Principle. — *If one variable varies inversely as a second, then the product of the variables is constant.*

136. Joint Variation. — One quantity varies *jointly* as two or more other quantities, when it varies directly as the product of those quantities. Thus the distance an automobile runs varies jointly as the rate and time, the area of a triangle varies jointly as the base and altitude, and the volume of a rectangular solid varies jointly as the length, width, and height.

In the illustration of the triangle just given

$$A \propto bh.$$

Then $$A = kbh.$$

Those of you who know geometry will recognize the fact that in this case the constant of variation, k, equals $\frac{1}{2}$.

137. Combined Variation. — In some examples one variable may be found to vary directly as another variable and inversely as a third variable. Thus, if x varies directly as y and inversely as z, then $x = \frac{ky}{z}$.

Illustration: The time required to dig a ditch varies directly as the length of the ditch in rods and inversely as the number of men working on the job. That is, $t = \frac{kr}{n}$.

Illustrative examples:

1. If x varies as y^2, and $x = 4$ when $y = 8$, find x when $y = 2$.

Solution: If x varies as y^2, then $x = ky^2$.

Then, if $x = 4$ and $y = 8$, $4 = k(8)^2$, and $k = \frac{1}{16}$.

Now, if $y = 2$, $x = ky^2$ becomes $x = \frac{1}{16}(2)^2$.

Therefore, $x = \frac{1}{4}$.

2. If x varies inversely as y, and $x = 3$ when $y = 8$, find x when $y = 6$.

Solution: $xy = k$, hence $3 \cdot 8 = k$, or $k = 24$.
Then, if $y = 6$, $6x = 24$, and $x = 4$.

3. If x varies jointly as y and z, and $x = 6$ when $y = 8$ and $z = 10$, find x when $y = 40$ and $z = 4$.

Solution: $x = kyx$. Hence, $6 = k \cdot 8 \cdot 10$.
Therefore, $k = \frac{3}{40}$.
Then, if $y = 40$ and $z = 4$, $x = \frac{3}{40} \cdot 40 \cdot 4$.
Therefore, $x = 12$.

4. If x varies directly as y and inversely as z, and $x = 3$ when $y = 5$ and $z = 10$, find x when $y = 4$ and $z = 2$.

Solution: $x = \dfrac{ky}{z}$, that is, $3 = \dfrac{k \cdot 5}{10}$. Hence, $k = 6$.

Now, if $y = 4$ and $z = 2$, $x = \dfrac{6 \cdot 4}{2}$, or $x = 12$.

5. The pressure of a gas varies jointly as its density and its absolute temperature. When the density is 1 and the temperature is 300° the pressure is 15. What is the pressure when the density is 3 and the temperature is 320°?

Solution: We have $P = kTD$.
Then, $15 = k \cdot 300 \cdot 1$. Hence, $k = \frac{1}{20}$.
When $D = 3$ and $T = 320$,
we obtain $P = \frac{1}{20} \cdot 320 \cdot 3 = 48$.

6. The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 2 inches in diameter weighs 2.5 pounds, find the weight of a sphere 4 inches in diameter.

Solution: $W \propto D^3$, then $W = kD^3$.
If $W = 2.5$ and $D = 2$, $2.5 = 8k$ and $k = \frac{5}{16}$.
Then, when $d = 4$, we have $W = \frac{5}{16} \cdot 4^3 = 20$ pounds.

Exercises

Express the relations indicated in the first ten examples both by use of the sign of variation and as an equation when k represents the constant.

1. x varies as y.

2. The area (A) of a circle varies directly as the square of its radius (r).

3. The surface of a sphere varies directly as the square of its diameter.

4. The circumference of a circle varies as the diameter.

5. The area of a rectangle varies jointly as the base and altitude.

6. The speed of a train in miles per hour varies inversely as the time if the distance is constant.

7. The weight of a body above the earth's surface varies inversely as the square of its distance from the center of the earth.

8. The apparent size of an object varies inversely as its distance from the observer. By "size" we mean any linear dimension.

9. The amount a bar will bend varies inversely as the width if the length and thickness remain the same.

10. The amount of illumination on a body varies directly as the intensity of the light and inversely as the square of the distance from the light.

11. If x varies directly as y, and $x = 75$ when $y = 5$, what is the value of x when $y = 3$?

12. If x varies inversely as y, and $x = 13$ when $y = 9$, what is the value of x when $y = 3$?

13. If x varies directly as y, and $x = 4$ when $y = 8$, find x when $y = 6$.

14. If x varies jointly as y and z, and equals 24 when $y = 2$ and $z = 3$, find x when $y = 3$ and $z = 5$.

15. If x varies jointly as y and z, and equals 40 when $y = 2$ and $z = 5$, find x when $y = 7$ and $z = 9$.

16. If x varies directly as y and inversely as z, and if $x = 60$ when $y = 4$ and $z = 6$, find x when $y = 5$ and $z = 30$.

17. If x varies inversely as the square of y, and if $x = 8$ when $y = 2$, find x when $y = 4$.

18. If x varies inversely as y, and x equals $\frac{3}{4}$ when $y = \frac{2}{3}$, what is the value of x when $y = \frac{5}{4}$?

19. If the square of x varies directly as the cube of y, and if $x = 2$ when $y = 4$, find the value of y when $x = 8$.

20. If $3x - 4$ varies directly as $y + 5$, and $x = 2$ when $y = 3$, find x when $y = 15$.

21. It has been found by experiment that the distance a body falls from rest varies directly as the square of the time. If a body falls 256 feet in 4 seconds, how far will it fall in 8 seconds?

22. The height of the Washington Monument is 555 feet. How long will it take a ball to fall to the ground from its top?

23. From what height was a stone dropped which took 12 seconds to reach the ground?

24. The volume of a sphere varies directly as the cube of its radius. If the volume of a sphere whose radius is 7 inches is 1437.3 cubic inches, what is the volume of a sphere whose radius is 5 inches?

25. The area of the curved surface of a right circular cylinder varies jointly as the radius and the altitude. If the area of such a surface whose radius is 12 inches and whose altitude is 10 inches is 240π square inches, what is the area of a similar surface whose radius is 8 inches and whose altitude is 9 inches?

26. The surface of a cube varies directly as the square of its edge. The surface of a cube whose edge is 6 inches is 216 square inches. What is the surface of a cube whose edge is 8 inches?

27. The volume of a right circular cylinder varies jointly as its altitude and the square of the radius of its base. If the volume of such a cylinder whose altitude is 10 and whose radius is 5 is $250\,\pi$, what is the volume of the cylinder if the altitude is 8 and the radius 8?

28. If $x \propto \frac{1}{\sqrt{y}}$ and $x = 4$ when $y = 36$, find the value of x in terms of y. Of y in terms of x.

29. The velocity of the rim of a pulley varies directly as its diameter. A 16-inch pulley has a rim velocity, at a certain moment, of 155 feet per minute. What is the rim velocity, at the same moment, of a $9\frac{1}{4}$-inch pulley on the same shaft?

30. Boyle's Law in chemistry states that under constant temperature the volume of a gas varies inversely as its pressure. If a gas has a volume of 106 cubic inches under a pressure of 14.7 pounds, find the volume when the pressure is 15.3 pounds.

31. The safe load of a horizontal beam 10 feet in length, supported at both ends, varies jointly as its breadth and the square of its depth. If a 2 by 6 pine joist of this length, standing on edge, will safely hold 800 pounds, how much will a 2 by 10 joist safely hold?

32. The amount of bending of a beam supported at both ends and loaded in the center varies directly as the load times the cube of the length in feet, and inversely as the width in inches times the cube of the depth in inches.

A beam 20 feet long, 5 inches wide, and 4 inches thick, supported at the ends and loaded at the center with 4000 pounds will bend downward $2\frac{1}{4}$ inches.

What must be the depth of a beam of the same material, length, and width, that shall bend only $\frac{3}{4}$ of an inch under the same load?

33. If x varies inversely as y, and y varies inversely as z, show that x varies as z.

Review

1. Determine, without solving, the character of the roots of

(a) $x^2 - 7x + 6 = 0$.

(b) $3x^2 + 5x - 2 = 0$.

(c) $2x^2 - x + 8 = 0$.

2. Write the equation whose roots are $3 + 5i$ and $3 - 5i$.

3. State the Factor Theorem. Use it to factor

$$x^3 + x^2 - 8x - 12.$$

4. Solve $(x^2 - 1)^2 - 3(x^2 - 1) = 10$.

5. Multiply $3x^{n+1} + 5x^n - 4$ by $6x^{n-1} + 7$.

6. Divide $y^6 - 2y^3 + 1$ by $y^2 - 2y + 1$.

7. Solve $x^{\frac{3}{2}} = -27$.

8. Solve the system $3y^2 + z^2 = 12$.

$$4y^2 - 5yz = -11.$$

9. Solve $x + \frac{1}{x} = 24$.

10. The diagonals of a rectangle are represented by $x^2 + 13x - 35$ and $2x^2 + x$. Find both values of x and the diagonals.

11. Factor

(a) $(x - a)^4 - 16a^4$.

(b) $(a^3 + 8b^3)(a + b) - 6ab(a^2 - 2ab + 4b^2)$.

(c) $x^6 - y^6$.

12. Solve for x and check:

(a) $\dfrac{3}{1-3x} + \dfrac{5}{1-5x} = \dfrac{4}{1-2x}$.

(b) $\dfrac{1}{(x+1)(x+4)} = \dfrac{1}{(x+2)(x+5)}$.

13. What is the reciprocal of $\dfrac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a} + \frac{1}{b}}$? By what must the given fraction be multiplied to produce unity?

14. Prove that $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$ and that $a^m \cdot a^{-n} = a^{m-n}$.

15. Find the value of $\dfrac{\sqrt{3} - \sqrt{5}}{\sqrt{3} + \sqrt{5}}$ correct to hundredths in the shortest manner.

16. Simplify: $16^{\frac{1}{4}}$, $16^{-\frac{1}{2}}$, 7^0, $(\frac{1}{3})^{-3}$, $49^{-\frac{3}{2}}$.

17. Which of the following processes is correct and which is incorrect?

$$\frac{3a^2}{x^2+x+1} = \frac{3a^2}{x^2} + \frac{3a^2}{x} + 3a^2.$$

$$\frac{x^2+x+1}{3a^2} = \frac{x^2}{3a^2} + \frac{x}{3a^2} + \frac{1}{3a^2}.$$

Prove your answer.

18. Find the side of a square if the area has the same number of square feet as the number of linear units in the perimeter.

19. A certain number decreased by three fifths of the square root of the number equals 22. Find the number.

20. The perimeter of an isosceles triangle is 18.9 inches. Each of the equal sides is 6 inches longer than the third side. What is the length of each side?

21. Prove that

$$(x-2)^2, x^2-4x+4, \text{ and } \frac{x^3-2x^2-4x+8}{x+2}$$

might represent the three sides of an equilateral triangle.

22. If 1 be added to both the numerator and the denominator of a fraction, its value is $\frac{1}{2}$; but if 1 be subtracted from both numerator and denominator, its value is $\frac{7}{16}$. Find the fraction.

Simplify:

23. $(x^{\frac{a}{a-1}})^{a^2-1} \div \frac{(x^{\frac{n}{2}})^2}{(x^2)^{\frac{1}{2}}}$.

24. $\frac{3^n \cdot (3^{n-1})^n}{3^{n+1} \cdot 3^{n-1} \cdot 9^{-n}}$.

25. Find the sum of $2+i$, $2-i$, $1+i$, and represent the sum graphically.

26. Solve the formula $2a^2 + 2b^2 = c^2 + 4m^2$ for the value of m.

Find the value of m, when $a = 12$, $b = 16$, and $c = 20$.

27. The product of the two larger of three consecutive odd numbers is the same as the sum of all of the numbers. Find the numbers.

28. Find the value of k in the following equation in order that the roots shall be equal:

$$x^2 - 2kx + 6x + 4k = 0.$$

29. One root of the equation $2x^2 - 15x + 7 = 0$, is .5. Find, without solving, the other root.

30. Solve $\sqrt{x+7} - \sqrt{5(x-2)} = 3$.

31. A line 17 inches long is divided in the ratio of 3 to 5. How long is each part?

32. Find the fourth proportional to

$$x+y,\ x-y,\ x^2+2xy+y^2.$$

33. Given the formula $S = \pi R^2 - \pi r^2$, and $R = 1.2$, $r = .8$, and $\pi = 3.1416$. Find in the shortest way the area of this circular ring.

34. Prove that a number of three digits is exactly divisible by 9 if the sum of the digits is exactly divisible by 9.

Hint: $N = 100\,h + 10\,t + u$

$= 99\,h + 9\,t + h + t + u.$

35. Prove that a number is exactly divisible by 3 if the sum of its digits is exactly divisible by 3.

CHAPTER XIV

SERIES OR PROGRESSIONS

We frequently meet in our study of the sciences and elsewhere, a succession of quantities which are related in a definite way to each other or to their numbers in the sequence. The present chapter treats of such sequences of related numbers and their applications.

138. Series. — A *series* or *progression* is a succession of terms each one of which is formed by the application of a fixed law. To classify a series, or to continue one to a greater number of terms, we must first discover the law upon which it is based.

Examples: $-3, 0, 3, 6, 9, \cdots$ is a series in which each term is formed from the preceding one by adding 3.

$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$ is a series in which each term is formed from the preceding one by multiplying it by $\frac{2}{3}$.

$1, 4, 9, 16, \cdots$ is a series in which the terms are formed by squaring the natural numbers in order.

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 \cdots$ is a series in which each term is formed by multiplying the number of the term in the series by the next succeeding number, etc.

Oral Exercises

By discovering its law, continue each of the following series to three additional terms.

1. $3, 1, -1, -3 \cdots$.
2. $x, x^2, x^3, x^4, \cdots$.
3. $y, -\frac{y}{2}, \frac{y}{4}, -\frac{y}{8}, \cdots$.

4. $1, 9, 25, 49, \cdots$.

5. $4, 16, 36, 64, \cdots$.

6. $1, \sqrt{2}, 2, 2\sqrt{2}, \cdots$.

7. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \cdots$.

8. $1 \cdot 3, 2 \cdot 4, 3 \cdot 5, 4 \cdot 6, \cdots$.

9. $\frac{x^2}{y^2}, \frac{x}{y}, 1, \frac{y}{x}, \cdots$.

10. $-5\sqrt{x}, -2\sqrt{x}, \sqrt{x}, 4\sqrt{x}, \cdots$.

11. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

12. $-\frac{3}{2}, 1, -\frac{2}{3}, \frac{4}{9}, \cdots$.

139. An **Arithmetical Progression** is a series in which each term is obtained from the preceding one by adding the same fixed quantity. This quantity may be either positive or negative, and is called the *common difference*. It may be found by subtracting any term from the one which follows it.

Select the arithmetical progressions from the above oral exercises.

In problems involving arithmetical progression we have to consider the following five quantities:

a = the first term,
l = the last term,
n = the number of the terms,
d = the common difference,
s = the sum of the n terms.

140. The Last Term. — In accordance with the above definitions any arithmetical progression can be represented generally thus:

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ a + 5d, \text{ etc.}$$

By inspection of these terms we notice that they differ only in the coefficient of d, which in the first term is 0, in the second 1, in the third 2, in the fourth 3, and so on. Evidently the coefficient of d is always one less than the number of the

term. Since the last term is the nth term, the coefficient of d in that term must therefore be $n - 1$. Examples:

The 6th term is $a + 5\,d$.
10th term is $a + 9\,d$.
nth term is $a + (n - 1)d$.

Hence the formula

$$l = a + (n - 1)d.$$

This formula may be said to be derived by observation.

141. Arithmetical Means. — The arithmetical mean between two numbers is that number which, standing between them, forms with them an arithmetical progression. Thus, if M is the arithmetical mean between a and b, then a, M, and b form an arithmetical progression. Then, since the difference between successive terms is constant,

$$M - a = b - M.$$

Then, $2\,M = a + b$ and $M = \frac{a + b}{2}$.

Hence the arithmetical mean between two numbers is *half their sum.*

If more than one mean is to be inserted, we must obtain the common difference and then insert the means. (See Illustrative example II.)

Illustrative example I:

Find the 8th term of the series 3, $\frac{3}{2}$, 0, etc.

Consider the 8th term the last term. Then

By formula

$$\begin{aligned} l &= a + (n - 1)d. \\ &= 3 + 7(-\tfrac{3}{2}). \\ &= 3 - \tfrac{21}{2}. \\ &= -\tfrac{15}{2}. \end{aligned}$$

Illustrative example II:

Insert 5 arithmetical means between 4 and 18.

There will be 7 terms in the progression after the means are inserted. Hence, $a = 4$, $n = 7$, $l = 18$.

$$l = a + (n - 1)d.$$
$$18 = 4 + 6\,d.$$
$$6\,d = 14.$$
$$d = 2\tfrac{1}{3}.$$

Hence, 4, $6\frac{1}{3}$, $8\frac{2}{3}$, 11, $13\frac{1}{3}$, $15\frac{2}{3}$, 18 is the series.

Illustrative example III:

Find the arithmetical progression in which the 4th term is 4 and the 7th term is -8.

Denote the 4th term by $a + 3\,d$, and the 7th term by $a + 6\,d$.

Then, $a + 6\,d = -8$

$a + 3\,d = 4$

Subtract, $3\,d = -12$, and $d = -4$.

Substituting, $a - 12 = 4$, and $a = 16$.

Hence the progression is 16, 12, 8, 4, etc.

Exercises

1. Write 4 terms of the arithmetical progression in which $a = -\frac{3}{2}$ and $d = \frac{5}{2}$; in which $a = x - 3\,y$ and $d = 2\,y$.

2. Write as an equation the fact that x, y, and z form an arithmetical progression.

3. Find the 8th term of the series $1, \frac{3}{4}, \frac{1}{2} \cdots$.

4. Find the 6th term of the series $2, 2\frac{3}{4}, 3\frac{1}{2}, \cdots$.

5. Find the 7th term of the series $2\,a - 5\,b$, $3\,a - 4\,b$, $4\,a - 3\,b, \cdots$.

6. Find the 12th term of the series $4\sqrt{3}, \sqrt{3}, -2\sqrt{3}, \cdots$.

7. Find the 5th term of the series $3 - \frac{1}{2}\sqrt{2}, 2, 1 + \frac{1}{4}\sqrt{2} \cdots$.

8. Find the 10th term of the series $5\sqrt{2}$, $\frac{14}{\sqrt{2}}$, $9\sqrt{2}$, $\cdots$.

9. Insert one arithmetical mean between 4 and $12\frac{1}{2}$, between -16 and -3, between $a-2\sqrt{3}$ and $a+2\sqrt{3}$, between h and k.

10. Insert 3 arithmetical means between 2 and 5.

11. Insert 4 arithmetical means between -62 and 3.

12. Insert 5 arithmetical means between 7 and 70.

13. Insert 5 arithmetical means between $\sqrt{a}+3$, and $25\sqrt{a}+27$.

14. Insert 7 arithmetical means between $\sqrt{3}-\sqrt{2}$ and $5\sqrt{3}+\frac{5}{3}\sqrt{2}$.

15. Insert 3 arithmetical means between m and n.

16. Find the first 3 terms of the arithmetical progression in which the third term is $3\frac{1}{4}$ and the 7th is $5\frac{1}{2}$.

17. The 5th term of an arithmetical progression is $7-11\sqrt{6}$, and the 8th term is $13-20\sqrt{6}$. Find the 12th term.

18. The product of two numbers is 54 and their arithmetical mean is $7\frac{1}{2}$. Find them.

Review

1. Simplify $8^{-\frac{2}{3}}+25^{\frac{1}{2}}+2^{-2}+6^{0}$.

2. What is the discriminant of a quadratic equation?

3. Solve by formula $x^2-2ax+a^2-b^2=0$.

4. Solve by substitution
$$3x+2y=15.$$
$$x-6y=15.$$

142. Sum of an Arithmetical Progression. — We shall now derive a formula for obtaining the sum of n terms of an arithmetical progression. Denote the required sum by S.

Then,

$$S = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (l - d) + l.$$

Reversing,

$$S = l + (l - d) + (l - 2d) + (l - 3d) + \cdots + (a + d) + a.$$

Adding,

$$2S = (a+l) + (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)$$

Hence $2S = n(a + l)$ and $S = \frac{n}{2}(a + l)$. (1)

Usually, however, we do not have the last term given in examples where the sum is required. A more useful formula for S in terms of a, n, and d may be found as follows:

In the formula $S = \frac{n}{2}(a + l)$, eliminate l by substituting for it $a + (n-1)d$, since we have learned that $l = a + (n-1)d$.

We thus obtain $S = \frac{n}{2}[2a + (n - 1)d]$. (2)

Illustrative example I:

Find the sum of 10 terms of the arithmetical progression in which $a = 3$, and $l = 63$.

Using formula (1) $S = 5(3 + 63) = 5 \cdot 66 = 330$.

Illustrative example II:

Find the sum of 16 terms of the series $\frac{2}{3}$, 1, $\frac{4}{3}$, $\cdots$.

Here $a = \frac{2}{3}$, $d = \frac{1}{3}$, $n = 16$. Using formula (2), we have, $S = 8(\frac{4}{3} + 15 \cdot \frac{1}{3}) = 8(\frac{4}{3} + 5) = 8 \cdot \frac{19}{3} = \frac{152}{3} = 50\frac{2}{3}$.

Exercises

Find the sum of the series in which

1. $a = -6$, $n = 12$, and $l = 72$.
2. $a = \frac{3}{52}$, $n = 20$, and $l = \frac{32}{52}$.
3. $a = 2x - 3y$, $n = 7$, and $l = 12x + 15y$.
4. Find the sum of 8 terms of $4\sqrt{3}$, $-\sqrt{3}$, $-6\sqrt{3}$, $\cdots$.

5. Find the sum of 18 terms of $2a - 5b$, $3a - 4b$, $4a - 3b, \cdots$.

6. Find the sum of 6 terms of $5\sqrt{2} + 1$, $7\sqrt{2} + 5$, $9\sqrt{2} + 9, \cdots$.

7. In the series $\frac{2}{3}$, 1, $\frac{4}{3}, \cdots$ which term is $25\frac{1}{3}$?

8. In the series of odd numbers 1, 3, 5, etc., how many terms must be taken that the sum may be 225?

9. Determine a formula for the sum of the first n even numbers.

10. In the series $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}, \cdots$ how many consecutive terms next preceding $\frac{3}{8}$ must be taken that their sum may be -15?

11. How many numbers of three figures each are divisible by 3? Find their sum.

12. How many numbers of four figures each are divisible by 7? Find their sum.

13. Find the sum of all numbers less than 10,000 which are divisible by 11.

14. If \$100 is placed on simple interest at 6%, it amounts in one year to \$106, in two years to \$112, etc. What will be the amount at the end of 24 years? In how long a time will it double itself?

15. A body falling from rest will fall 16 feet the first second, and thereafter will fall 32 feet farther each second than it fell the preceding one. How far will it fall the 5th second? How far will it fall in 5 seconds?

16. How many seconds will it take a body to fall from the top of the Washington Monument, which is 555 feet high?

17. In rolling down an incline a body rolled 5 feet the first second, and increased its velocity 10 feet per second. How far will it roll in 8 seconds?

18. A body was shot upward at a rate which carried it 1000 feet the first second and its velocity thereafter decreased at the rate of 32 feet each second. How high was the projectile at the end of 12 seconds? How many feet upward did it go during the 12th second? How far during the 40th second? What is the meaning of the negative answer in the latter case?

19. A slide in an amusement park, built on a uniform grade, is supported by 41 upright posts, the shortest being 1 foot long, and the longest 61 feet long. Find the total length of these posts, if equally distant apart.

20. A teacher began her services at a salary of \$1600 a year, and received an annual increment of \$100 for eight consecutive years thereafter, after which her salary remained stationary. How much did she earn in 30 years of teaching?

21. The digits of a certain number of 3 digits are in arithmetical progression. Their sum is 15, and the number is 129 times its first digit. What is the number?

22. Find 3 numbers in arithmetical progression whose sum is 6 and the sum of whose squares is 62.

Hint: Let $x - y$ denote the first term, and y the common difference.

23. In a certain arithmetical progression whose first term is 3, the 9th term is to the 4th as the 16th is to the 7th. Find the series.

24. Two boys run a race of $\frac{1}{4}$ mile. The first runs at a uniform rate of 1 rod in 1 second. The other runs the first rod at the rate of 4 rods in 3 seconds, but decreases his speed uniformly by taking $\frac{1}{100}$ of a second longer to run each rod than it took to run the preceding one. Which boy will win the race and by how many seconds?

Review

1. Solve graphically $3x^2 - x - 10 = 0$.

2. A freight train ran from A to B in 5 hours. If it had run 5 miles an hour faster, it would have made the trip in 1 hour less time. What is the distance from A to B?

3. Divide $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{6}} + y^{\frac{1}{6}}$.

4. Find the square root of $\frac{a^4}{4} - 3a + \frac{13a^2}{4} + 9 - \frac{a^3}{2}$.

5. Derive the quadratic formula.

Geometrical Progression

143. A Geometrical Progression is a series of terms each of which is formed by multiplying the one before it by the same fixed quantity. This quantity is called the *common ratio*. It may be found by dividing any term by the preceding one. The common ratio may be integral or fractional, negative or positive.

Select the geometrical progressions from the series listed under the oral exercises in section 138.

In formulas concerning geometrical progressions the common ratio is denoted by r. The letters a, l, n, and S have the same meaning as in an arithmetical progression.

144. Last Term of a Geometrical Progression. — In accordance with the above explanation the general representation of a geometrical progression is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots\cdot$$

We notice that these terms are identical except for the exponent of r, which is 0 in the first term, 1 in the second, 2 in the third, etc. This exponent is evidently one less than the number of the term in the series, in each case. Since

the last term is the nth term, the exponent of r in that term must be $n - 1$. Thus, 7th term is ar^6

10th term is ar^9

nth term is ar^{n-1}

Hence the formula $\quad l = ar^{n-1}$.

This formula, like the corresponding formula in an arithmetical progression, is derived by observation.

145. Geometrical Means. — A geometrical mean between two quantities is that quantity which, standing between them, forms with them a geometrical progression.

Let the geometrical mean between a and b be denoted by M. Then a, M, b forms a geometrical progression. Then, since there is a constant ratio between successive terms,

$$\frac{M}{a} = \frac{b}{M}.$$

Therefore, $\quad M^2 = ab$

and $\quad M = \pm\sqrt{ab}$.

Hence the geometrical mean between two numbers is the square root of their product. What have we called this expression in a former chapter?

If more than 1 mean is to be inserted, we must first obtain the common ratio. (See Illustrative example II below.)

Illustrative example I:

Find the 5th term of the geometrical progression $\frac{2}{3}$, 1, $\frac{3}{2}$, $\cdots$.

Here $a = \frac{2}{3}$, $r = \frac{3}{2}$, $n = 5$.

By formula, $\quad l = ar^{n-1}$.

Then, $\quad l = \frac{2}{3}(\frac{3}{2})^4$.

$$l = \tfrac{2}{3} \cdot \tfrac{81}{16}.$$

$$l = \tfrac{27}{8}.$$

Illustrative example II:

Insert 4 geometrical means between $3a^2$ and $\frac{96}{a^3}$.

Here $a = 3a^2$, $l = \frac{96}{a^3}$, $n = 6$, and we are to find r.

Since $$l = ar^{n-1},$$

$$\frac{96}{a^3} = 3a^2(r)^5.$$

Dividing both members by $3a^2$, $r^5 = \frac{32}{a^5}$.

Hence, $$r = \frac{2}{a}.$$

Then, inserting the means, we obtain

$$3a^2,\ 6a,\ 12,\ \frac{24}{a},\ \frac{48}{a^2},\ \frac{96}{a^3}.$$

Illustrative example III:

Determine the geometrical progression in which the third term is 2 and the 6th term is $\frac{27}{32}$.

The third term is represented by ar^2, the 6th term by ar^5.

Then, $$ar^5 = \tfrac{27}{32}. \quad (1)$$

$$ar^2 = 2. \quad (2)$$

Divide (1) by (2), then $r^3 = \frac{27}{64}$, and $r = \frac{3}{4}$.

By substituting in (2), $a = \frac{32}{9}$.

Hence we have $\frac{32}{9}, \frac{8}{3}, 2, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}, \cdots$.

Exercises

1. Write 5 terms of the geometrical progression whose first term is $-\frac{1}{2}$, and the common ratio -4.

2. Test the series $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \cdots$ to determine whether it is arithmetical or geometrical.

3. Write as an equation the fact that x, y, and z are in geometrical progression.

4. Find the 6th term of the series $2, 2\sqrt{2}, 4, \cdots$.

5. Find the 7th term of the series $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \cdots$.

6. Find the 8th term of the series $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \cdots$.

7. Find the 6th term of the series $2+\sqrt{3}, 1, 2-\sqrt{3}, \cdots$.

8. Find the 10th term of the series $a^0, a^{\frac{4}{3}}, a^{\frac{8}{3}}, \cdots$.

9. Insert 6 geometrical means between $\frac{3}{2}$ and 192.

10. Insert 4 geometrical means between -4 and 972.

11. Insert 3 geometrical means between $\frac{1}{6}$ and $\frac{8}{243}$.

12. Insert 5 geometrical means between $\frac{x^2}{y^8}$ and $\frac{y^4}{x^4}$.

13. Insert 4 geometrical means between $3\sqrt{2}$ and $27\sqrt{6}$.

14. Insert 2 geometrical means between m and n.

15. If the third term of a geometrical progression is $\frac{1}{12}$ and the 8th term is $\frac{8}{3}$, find the first two terms.

16. The second term of a geometrical progression is $3a^2$ and the 6th term is $\dfrac{243}{16\,a^4}$. Find the 8th term.

Review

1. Solve for x, $\dfrac{1}{2}\left(x-\dfrac{a}{3}\right)-\dfrac{1}{3}\left(x-\dfrac{a}{4}\right)+\dfrac{1}{4}\left(x-\dfrac{a}{5}\right)=0$.

2. Factor $x^2-2x+1-a^2-4b^2-4ab$.

3. A room is a feet long and b feet wide. In the center is a rug c feet square. How many square yards of linoleum will be required to cover the rest of the floor?

146. Sum of n Terms of a Geometrical Progression. — If we denote the sum by S, we have

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}.$$

Then, $$rS = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n,$$

by multiplying both members of the equation by r.

Subtracting, $S - rS = a - ar^n$.

$$S(1 - r) = a - ar^n.$$

$$S = \frac{a - ar^n}{1 - r} \text{ or } \frac{ar^n - a}{r - 1}.$$

A second formula for S may be derived thus. Since $l = ar^{n-1}$, we may multiply both members of the equation by r, and obtain $rl = ar^n$. Now if, in the above formula, we substitute for ar^n its equal rl, we obtain the formula

$$S = \frac{a - rl}{1 - r} \text{ or } \frac{rl - a}{r - 1}.$$

Exercises

1. Find the sum of 8 terms of the series $\frac{1}{2}$, 1, 2 $\cdots$.

2. Find the sum of 6 terms of the series 3, $-$ 6, 12, $\cdots$.

3. Find the sum of 5 terms of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, $\cdots$.

4. Find the sum of 12 terms of the series 48, 24, 12, $\cdots$.

5. Find the sum of 7 terms of the series $3\sqrt{2}$, 12, $24\sqrt{2}$, $\cdots$.

6. Find the sum of 6 terms of the series $\sqrt{2} + 1$, 1, $\sqrt{2} - 1$, $\cdots$.

7. How many ancestors have you had, counting back ten generations?

8. In the series $\frac{2}{3}$, 2, 6, $\cdots$, which term is 486?

9. To obtain the amount of \$50, bearing interest at 6%, at the end of one year, we may multiply \$50 by 1.06. If the interest is not paid, it may be added to the principal, so that \$53 bears interest the second year. Then we multiply \$53 by 1.06 to obtain the amount at the end of the second year, etc. This method of computing interest is called compound interest compounded annually. The original principal and the amounts at the ends of successive years form a geometrical progression in which the common ratio is 1.06.

What will be the amount of \$50, so invested, at the end of 4 years?

10. Find the amount of \$840, bearing compound interest at 4%, at the end of 5 years?

11. If the value of an automobile depreciates each year 40% of its value the preceding year, what is the value at the end of 4 years of a car which cost \$2400?

12. A blacksmith agreed to shoe a horse on condition that he was to receive 1 cent for driving the first nail, 2 cents for the second, 4 cents for the third, etc. How much did he receive for shoeing the horse? (Old arithmetical puzzle.)

NOTE. A horse's shoe contains 8 nails.

13. A pail containing two quarts of water was passed in succession to 5 workmen. Each drank $\frac{1}{4}$ of the water which was in the pail when he received it. How much water was left in the pail after all had drunk?

14. Write a succession of terms which may be considered a special case of either an arithmetical progression or of a geometrical progression.

Review

1. Write the equation whose roots are $3 - i$ and $3 + i$.

2. When a lever with a weight on each end is balanced over a fulcrum, the weights and their respective distances from the fulcrum are inversely proportional. Using W and W' to represent the weights, and D and D' their respective distances from the fulcrum, write the formula connecting them.

3. Without solving, determine whether $1+\sqrt{5}$ and $1-\sqrt{5}$ are roots of the equation $x^2 - 2x - 4 = 0$.

4. What values of x will make $x^2 - x - 4$ equal to $2x^2 + 3x - 9$?

147. Infinite Geometrical Progression. — The following problem illustrates a type of geometrical progression which deserves special consideration.

Suppose a point P starts to move from A toward B, 2 feet distant from A, going one half the distance, or 1 foot, the first second; one half the *remaining* distance, or $\frac{1}{2}$ foot, the second second; one half the *remaining* distance, or $\frac{1}{4}$ foot, the third second, and so on.

We may represent by a graph the distance the point P moves each second, thus

The distances traveled in successive seconds are evidently

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \text{etc.}$$

Now notice

1st. These numbers form a geometrical progression in which $r = \frac{1}{2}$.

2d. Each term is smaller than the preceding term.

3d. As we write more and more terms the last term written is approaching nearer and nearer to 0.

4th. The sum of n terms is approaching nearer and nearer to 2.

5th. The sum will never reach 2, and therefore will always be less than 2.

There is no number less than 2 which the sum will always be less than, if the number of terms is made larger and larger, hence 2 is called the *sum* of the series.

We shall now develop a formula for the sum of such a series as that just illustrated.

We have the formula

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Since r is less than 1, r^n is less than r. In fact, as n becomes larger and larger, the value of r^n becomes smaller and smaller, and may be made to approach indefinitely near to 0 by taking a sufficiently great number of terms. Hence the values of ar^n, and of the fraction $\frac{ar^n}{1-r}$ also approach indefinitely near to 0. As they do so, the value of S approaches indefinitely near to $\frac{a}{1-r}$. We therefore say that the sum of an infinite number of terms, or $S_\infty = \frac{a}{1-r}$.

You should clearly understand that the above formula gives the *limit* of the sum of an *indefinite* number of terms. If the sum of a *definite* number of terms is desired, we use the formula $S = \frac{a - ar^n}{1-r}$.

148. Repeating Decimals. — If we change a common fraction to decimal form, we shall find that the resulting decimal either comes out evenly after a certain number of figures, or else that the figures repeat in groups indefinitely. The repeating group is called the *repetend*.

Thus,

$$\tfrac{1}{3} = .333333333 \cdots.$$
$$\tfrac{4}{11} = .36363636 \cdots.$$
$$\tfrac{3}{7} = .428571428571 \cdots.$$
$$\tfrac{5}{6} = .833333333 \cdots.$$
$$\tfrac{797}{3300} = .24151515 \cdots.$$

In the first example the repetend is 3, in the second 36, in the third 428571, in the fourth 3, in the fifth 15. When the decimal repeats from the beginning, as in the first three examples, it is called a *pure repeating decimal*. When there are certain figures which do not repeat followed by repeating groups, as in the last two examples, it is called a *mixed repeat-*

ing decimal. Repeating decimals are also called *infinite decimals, recurring decimals,* and *circulating decimals.*

Repeating decimals furnish an excellent example of an infinite decreasing geometrical progression. For example, $.33333333 \cdots$ equals the sum of the terms $.3 + .03 + .003 + .0003 + .00003 + \cdots$, in which the common ratio is .1.

$$S_\infty = \frac{a}{1 - r}.$$

Hence, $$S_\infty = \frac{.3}{1 - .1} = \frac{.3}{.9} = \frac{3}{9} = \frac{1}{3}.$$

That is to say, the value of this decimal, if carried out indefinitely, approaches $\frac{1}{3}$ as a limit. Check by division.

Also $.363636 \cdots$ equals the sum of the series $.36 + .0036 + .000036 \cdots$, in which the common ratio is .01.

Then, $$S_\infty = \frac{.36}{1 - .01} = \frac{.36}{.99} = \frac{36}{99} = \frac{4}{11}.$$

The common ratio may be determined from the number of figures in the repetend. If one figure repeats, the ratio is .1; if two figures repeat, it is .01; if three figures repeat, it is .001, etc.

In the case of a mixed repeating decimal such as the fourth above, we have $.83333333 \cdots = .8 + .03 + .003 + .0003 + \cdots$. Here the series begins with .03 and the common ratio is .1.

Then, $$S_\infty = \frac{.03}{1 - .1} = \frac{.03}{.90} = \frac{1}{30}.$$

We now add the first term .8, which is not a part of the series.

$$\tfrac{8}{10} + \tfrac{1}{30} = \tfrac{25}{30} = \tfrac{5}{6}.$$

This is the sum to infinity.

Exercises

Find the sum of an infinite number of terms of each of the following series:

1. $2, \frac{4}{3}, \frac{8}{9}, \cdots$.
2. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots$.
3. $5, \frac{5}{3}, \frac{5}{9}, \cdots$.
4. $x, 1, \frac{1}{x}, \frac{1}{x^2}, \cdots$, where x is greater than 1.
5. $\frac{1}{2}, \frac{1}{4}\sqrt{2}, \frac{1}{4}, \cdots$.
6. $.44444444 \cdots$.
7. $.272727 \cdots$.
8. $.848484 \cdots$.
9. $.309309 \cdots$.
10. $.423423 \cdots$.
11. $.714825714825 \cdots$.
12. $.233333 \cdots$.
13. $.566666 \cdots$.
14. $.24545 \cdots$.
15. $.162222 \cdots$.
16. $.84515151 \cdots$.

17. If a rubber ball is dropped from a height of 4 feet, rebounds $\frac{1}{3}$ of that distance, and continues to rebound after each descent $\frac{1}{3}$ of the distance it dropped, over what distance will it have passed after 10 descents? Over what distance if it continues indefinitely?

Review

1. Can you solve the system $\begin{matrix} 4x - y = 3 \\ 8x = 2y - 5 \end{matrix}$? Why?
2. Prove that $a^0 = 1$, regardless of the value of a.
3. A room is 20 feet long, 15 feet wide, and 12 feet high. Find to the nearest tenth of a foot the length of a diagonal drawn from an upper corner to the opposite lower corner.
   Hint: $d^2 = l^2 + w^2 + h^2$.
4. Simplify:
$$\frac{b+y}{a^3+y^3} \cdot \frac{b}{c} \cdot \frac{ac+cy}{b^2+by} \div \frac{b^2+y^2}{b^6+y^6}.$$
5. Factor:
   (a) $21a^2 - 17a - 30$.
   (b) $(a^2 - 12)^2 - a^2$.

CHAPTER XV

THE BINOMIAL THEOREM

By rules already stated in our chapter on Products and Factors, we can square or cube mentally any binomial. It is frequently necessary, however, to raise a binomial to a power higher than the third, and it is desirable that we have a method of doing this shorter than multiplication. The method discussed in this chapter is also important historically, and in its application to higher mathematics.

149. The Binomial Theorem. — The Binomial Theorem is a method by means of which any binomial may be raised to any power. It is stated in formula form as follows.

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + \frac{n(n-1)\cdot\cdot\cdot 1}{1\cdot 2\cdot\cdot\cdot n}a^0b^n.$$

150. Proof by Mathematical Induction.* — This theorem

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* You should thoroughly memorize this theorem or formula. You will not find it difficult if you note the following points.

Exponents of a begin with n, which is the exponent that indicates the power to which the binomial is to be raised, and decrease by 1 in each term until a^0 is reached in the last term. Exponents of b begin with b^1 in the second term, and increase by 1 in each term until b^n is reached in the last term. In every term the sum of the exponents of a and b is n.

The coefficient of the second term is n, which we write as $\frac{n}{1}$. In successive terms there are always the same number of factors in the numerator of the coefficient as in its denominator, the coefficient of the third term containing 2 factors, the fourth term 3, and so on. In the numerator the factors begin with n and descend until 1 is reached in the last term. In the denominator the factors begin with 1 and ascend until n is reached in the last term.

is true for every value of n, but the proof for the general case is beyond the range of this book. When n represents any positive integer, it may be proved by a method which is called *mathematical induction*. The proof follows.

We know that

$$(a + b)^2 = a^2 + \frac{2}{1}ab + \frac{2 \cdot 1}{1 \cdot 2}b^2.$$

and $$(a + b)^3 = a^3 + \frac{3}{1}a^2b + \frac{3 \cdot 2}{1 \cdot 2}ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}b^3.$$

Hence these two expressions do follow the form stated in the general theorem. Now let n represent any value of the exponent for which the formula is known to be true, as, for example, 3. Then for that value of n,

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots.$$

Now multiply both members of this identity by $a + b$.

$$\begin{array}{lll} (a + b)^n & = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots & \\ a + b & \quad a + b & \\ \hline (a + b)^{n+1} & = a^{n+1} + \frac{n}{1}a^nb + \frac{n(n-1)}{1 \cdot 2}a^{n-1}b^2 + \cdots & \\ & + \quad a^nb + \quad \frac{n}{1}a^{n-1}b^2 + \cdots & \\ \hline (a + b)^{n+1} & = a^{n+1} + \frac{n+1}{1}a^nb + \frac{(n+1)n}{1 \cdot 2}a^{n-1}b^2 + \cdots & \end{array}.$$

Since this result is of the same form with reference to the exponent $n + 1$, that the original formula is with reference to the exponent n, it follows that, if this is a true form for the exponent n, it is also a true form for the exponent $n + 1$. In other words, if the theorem is true for *any* exponent, it is true for the *next higher* exponent. But it is true for exponent

3, as has already been shown. Hence it is true for exponent 4. Being true for 4, it is true for 5, and so on indefinitely. Hence the theorem may be proved true for any positive integral exponent.

151. Application of the Binomial Theorem. — When the exponent n is a positive integer, the number of terms in the series is $n + 1$.

Illustrative example I :

Expand by the Binomial Theorem $(2c - 3x^{\frac{1}{2}})^5$.

Here $a = 2c$, $b = -3x^{\frac{1}{2}}$, $n = 5$.

Substituting these values in the above formula, we obtain

$$(2c - 3x^{\frac{1}{2}})^5 = (2c)^5 + \frac{5}{1}(2c)^4(-3x^{\frac{1}{2}}) + \frac{5 \cdot 4}{1 \cdot 2}(2c)^3(-3x^{\frac{1}{2}})^2$$

$$+ \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(2c)^2(-3x^{\frac{1}{2}})^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(2c)(-3x^{\frac{1}{2}})^4$$

$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(2c)^0(-3x^{\frac{1}{2}})^5.$$

Note that the coefficient in the last term always becomes 1, that a (here $2c$) has disappeared since $(2c)^0 = 1$, and that b (here $-3x^{\frac{1}{2}}$), has reached the 5th power, the same as the power to which the binomial was to be raised.

Now, simplifying, we have

$$(2c - 3x^{\frac{1}{2}})^5 = 32c^5 - 240c^4x^{\frac{1}{2}} + 710c^3x - 1080c^2x^{\frac{1}{2}} + 810cx^2 - 243x^{\frac{5}{2}}.$$

In expanding by the Binomial Theorem it is always best to substitute first in the formula, and then to simplify as in the above example, rather than to try to combine the two processes.

Illustrative example II:

Expand to 4 terms $(x^{\frac{2}{3}} - \frac{1}{2})^{-3}$.

There is an infinite number of terms in the expansion when n is not a positive integer, hence there is no *last* term.

Here $a = x$, $b = (-\frac{1}{2})$, $n = -3$.

Substituting in the Binomial Formula, we have

$$\left(x^{\frac{2}{3}} - \frac{1}{2}\right)^{-3} = (x^{\frac{2}{3}})^{-3} + \frac{-3}{1}(x^{\frac{2}{3}})^{-4}\left(-\frac{1}{2}\right) + \frac{(-3)(-4)}{1 \cdot 2}(x^{\frac{2}{3}})^{-5}\left(-\frac{1}{2}\right)^2 + \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3}(x^{\frac{2}{3}})^{-6}\left(-\frac{1}{2}\right)^3 + \cdots.$$

Simplifying, we obtain

$$(x^{\frac{2}{3}} - \tfrac{1}{2})^{-3} = x^{-2} + \tfrac{3}{2}x^{-\frac{8}{3}} + \tfrac{3}{2}x^{-\frac{10}{3}} + \tfrac{5}{4}x^{-4} + \cdots,$$

or, if written with positive exponents,

$$= \frac{1}{x^2} + \frac{3}{2x^{\frac{8}{3}}} + \frac{3}{2x^{\frac{10}{3}}} + \frac{5}{4x^4} + \cdots.$$

152. *R*th, or General Term, of a Binomial Expansion. — Let r represent the *number* of any term which it is desired to write. We shall obtain a formula for this term by noticing how the form of any term is related to the number of that term in the series.

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots.$$

We note that in each term the coefficient contains the same number of factors in numerator and denominator, and that number is one less than the number of the term. Hence in the rth term the denominator is $1 \cdot 2 \cdot 3 \cdots (r-1)$. The exponent of b is the same as the last factor of the de-

nominator, and the exponent of a is n minus that same number, so that in any term the sum of the exponents of a and b is n. Finally the numerator of the coefficient begins with n, and goes down to a number which is one larger than the exponent of a.

Hence,

$$R\text{th term} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)} a^{n-r+1}b^{r-1}.$$

Illustrative example:

Find the 9th term in the expansion of $\left(x^2+\frac{1}{x}\right)^{-2}$.

Here $a = x^2$, $b = \frac{1}{x}$, $n = -2$, $r = 9$.

By the formula, 9th term

$$= \frac{(-2)(-3)(-4)(-5)(-6)(-7)(-8)(-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}(x^2)^{-10}\left(\frac{1}{x}\right)^8$$

$$= \frac{9}{x^{28}}.$$

Exercises

Expand completely:

1. $(a-3b)^3$.
2. $(2x^2-1)^5$.
3. $\left(\frac{x}{2}+\frac{2}{x^2}\right)^5$.
4. $\left(\sqrt{x}-\frac{3}{2x}\right)^6$.
5. $(m^{\frac{1}{2}}+2m)^5$.
6. $(1+3x^{\frac{2}{3}})^6$.
7. $(x^{-\frac{1}{2}}+2y^{\frac{1}{3}})^6$.

Expand to 4 terms:

8. $(1+x)^{-1}$.
9. $\left(\frac{x^2}{a}-a\right)^{-2}$.
10. $\left(x^{\frac{2}{3}}+\frac{1}{x}\right)^{-3}$.
11. $(1-x)^{\frac{1}{2}}$.
12. $(a^2-x^{-3})^{\frac{1}{2}}$.
13. $\left(x-\frac{\sqrt{y}}{2}\right)^{-\frac{1}{2}}$.
14. $\left(\frac{x}{2}+\frac{2}{y}\right)^{-2}$.

15. Write the 8th term in the expansion of $\left(1 - \frac{x^2}{2}\right)^{10}$.

16. Write the 6th term in the expansion of $\left(2 + \frac{a^2}{2}\right)^{20}$.

17. Write the 10th term in the expansion of $(a + \sqrt{b})^{-1}$.

18. Write the 8th term in the expansion of $\left(2 + \frac{1}{2\,a^{-1}}\right)^{-3}$.

19. Write the middle term in the expansion of $(x^2 - 6\,y^{\frac{3}{2}})^8$.

20. Write the term which does not contain x in the expansion of $\left(x^3 + \frac{3}{x^2}\right)^{15}$.

21. By use of the Binomial Theorem extract the cube root of 7 to three decimal places.

Hint: $\sqrt[3]{7} = 7^{\frac{1}{3}} = (8 - 1)^{\frac{1}{3}}$. After expanding, change the terms to decimal form and combine them.

22. Extract to three decimal places the fifth root of 30.

23. Extract to three decimal places the fourth root of 18.

24. The amount at compound interest of \$1000 for 5 years at 6% may be found by the formula $A = 1000(1.06)^5$. Compute the amount, writing $(1.06)^5$ as $(1 + \frac{3}{50})^5$ and expanding by the Binomial Theorem.

Review

1. Combine

$$\frac{x + 2}{x^2 + 3\,x - 40} - \frac{y - 2}{xy - 5\,y + 3\,x - 15}$$

and check when $x = 2$ and $y = 3$.

2. Simplify $\{-(x^3)^{\frac{1}{2}}\}^{\frac{1}{3}} \cdot (4\,x^{-3})^{\frac{1}{2}}$.

3. Solve $\frac{7\,y + 9}{4} - \left(y - \frac{2\,y - 1}{9}\right) = 7$.

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4. The three angles of a triangle are to each other as $2 : 3 : 10$. Find the number of degrees in each angle of the triangle, if their sum is $180°$.

5. Simplify $\dfrac{2}{1-\dfrac{1}{a^2}} - \dfrac{1}{1+\dfrac{1}{a}} - \dfrac{1}{a-\dfrac{1}{a}}$.

6. Divide $x^3 + y^3 - 3xy + 1$ by $1 + y + x$.

7. Find the Highest Common Factor and Lowest Common Multiple of $x^2 + 3x - 18$, $x^3 - 27$, and $x^2 - 6x + 9$.

8. Graph $x^2 - 8y^2 = 100$.

HISTORICAL NOTE

Sir Isaac Newton, who lived in England from 1642 to 1727, was perhaps the greatest mathematician and scientist who ever lived. His principal discoveries are contained in a monumental work called the "Principia," which was published by him in 1687. Among these discoveries we find the Binomial Theorem, the beginnings of the Calculus, the statement of the laws of motion, the fact of universal gravitation and the laws governing it. Newton also introduced the use of literal exponents, discovered a method of determining limits to the number of positive and negative roots in an equation, proved that imaginary roots occur in pairs in an equation with real coefficients, explained the decomposition of light and the cause of the rainbow, and invented the sextant and the reflecting telescope. He invented a theory concerning the nature of light known as the "Corpuscular Theory."

The last years of his life were marred by poor health and shattered nerves. He is buried in Westminster Abbey, where a great monument has been erected to his memory.

While Newton probably discovered that this theorem was true for positive integers as well as for negative and fractional values, Euler (1707–1783) was perhaps the first to give us a proof of the theorem, and Abel (1802–1829), a Norwegian, is probably the one to whom we are indebted for the first satisfactory proof.

CHAPTER XVI

LOGARITHMS AND TRIGONOMETRY

In solving mathematical problems much time is consumed in performing arithmetical operations which are often long and tedious. This chapter will explain the use of logarithms, by whose use the amount of labor involved in such operations is greatly lessened. They are here presented in an elementary way, and yet with sufficient fullness to make them very useful. In your study of advanced algebra and trigonometry a more complete discussion of logarithms will be given. Some simple problems of trigonometry are taken up in this chapter as an application of the use of logarithms.

153. Powers of 2. — The following table shows the powers of 2 up to the twentieth power. Suppose we wish to multiply 16 by 512. We note from the table that $16 = 2^4$ and $512 = 2^9$. Therefore, $16 \cdot 512 = 2^4 \cdot 2^9 = 2^{13}$, and this is shown by the table to be 8192. Hence we have multiplied 512 by 16 by merely adding exponents and looking up a corresponding number in the table.

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$
$2^{11} = 2048$
$2^{12} = 4096$

154. Logarithms. — In the statement $2^9 = 512$, 2 is the *base*, 9 is the *exponent*, and 512 is the resulting *number*. The exponent 9 may also be called the *logarithm* of 512 to base 2. This may be written $\log_2 512 = 9$.

$2^{13} = 8192$
$2^{14} = 16384$
$2^{15} = 32768$
$2^{16} = 65536$
$2^{17} = 131072$
$2^{18} = 262144$
$2^{19} = 524288$
$2^{20} = 1048576$

You should memorize the following definition.

The logarithm of a number to a given base is the exponent of the power to which the base must be raised that it may equal the given number.

From the above definition and illustration you will see that the statements $a^x = m$, and $\log_a m = x$, mean exactly the same thing.

155. Common Logarithms. — The base usually used in work with logarithms is 10, and logarithms computed on that base are called *common logarithms*. The table on pages 370, 371 is a table of common logarithms. When no other base is stated it is understood that the base is 10.

156. Characteristic and Mantissa. — Since most numbers are not exact powers of 10, their logarithms to base 10 are irrational numbers. Thus, since $10^1 = 10$, $\log 10 = 1$. Since $10^2 = 100$, $\log 100 = 2$. Since 73 is a number between 10 and 100, log 73 is a number between 1 and 2. As a matter of fact, it is 1.8633 correct to the nearest figure in the fourth decimal place. A table like the one in our book which gives the logarithm to the nearest ten-thousandth is called a *four-place table.*

The *characteristic* of a logarithm is the number which precedes the decimal point. It may be either positive or negative, or zero.

The *mantissa* is the decimal part of a logarithm. It is always positive .

Thus, in the logarithm of 73, the characteristic is 1 and the mantissa is .8633.

The table of logarithms gives only mantissas, since the characteristic can be inserted by the following

Rules. — (1) *If the number whose logarithm is desired is greater than 1, the characteristic is positive, and numerically one less than the number of figures to the left of the decimal point in the given number.*

(2) *If the number whose logarithm is desired is less than one, write it in decimal form. The characteristic is then negative and numerically one greater than the number of zeros immediately following the decimal point.*

When the characteristic is negative the $-$ sign should not be written in front of the logarithm, since it does not apply to the logarithm as a whole, but to the characteristic only. Instead we add 10 to the negative characteristic and subtract it again at the end of the logarithm. For example, the characteristic -1 should be written 9. mantissa -10, the characteristic -2 should be written 8. mantissa -10, etc. Example, $\log .03 = 8.4771 - 10$.

From the above rules and illustrations you will see that the characteristic depends entirely on the position of the decimal point in the given number. The mantissa, on the other hand, is entirely independent of the position of the decimal point, but depends solely on the arrangement of the figures in the given number. Thus the mantissas of the logarithms of 72,350, 7235, 72.35, .7235, and .0007235 are exactly the same, while their characteristics are 4, 3, 1, -1, and -4, respectively.

157. The Direct Use of the Tables. — The direct use of the table of logarithms consists in finding the logarithm of a given number. For example, let it be required to find the logarithm of 76.8. Since there are two figures to the left of the decimal point, the characteristic is 1. To find the mantissa we look for the first two figures of the given number in the left-hand column of the table, headed N. We find

76 near the middle of that column on the second page of the table. We then follow the horizontal line to the right until we come to the column headed 8 at the top. Here we read the mantissa .8854. Hence log 76.8 = 1.8854.

158. Interpolation. — If the given number contains more than three figures, we cannot find the logarithm directly from the table. As an illustration let us find the logarithm of 1864. The characteristic is 3. We shall, for the moment, consider the decimal point as after the third figure, thus 186.4. Doing so will not affect the mantissa since that is independent of the position of the decimal point. Now, as in the preceding example, we look up the mantissa of 186 and obtain .2695. We then look up the mantissa of 187 and obtain .2718. There is a difference between these mantissas of .0023. Now since the given number is .4 of the way between 186 and 187 we assume that the corresponding mantissa will be .4 of the way between .2695 and .2718. We therefore multiply .0023 by .4, carrying the result to the nearest figure in the fourth decimal place, and obtain .0009. This is added to the mantissa of 186 as a correction, giving the desired mantissa as .2704. Hence log 1864 = 3.2704.

This process of inserting a mantissa for 186.4 between the mantissas of 186 and 187 is called *interpolation*. The method used suggests that the logarithm varies as the number. While this is not true as a general principle, it is nearly enough true over a small interval so that the above process gives the correct result.

Illustrative example:

Find the logarithm of .93775. The characteristic is − 1, which we write in the form 9. . . . 10. The mantissa corresponding to the figures 937 is .9717, while that corresponding to 938 is .9722. Multiplying the difference .0005

by .75, we obtain .0004 to the nearest ten-thousandth. Adding this to .9717, we obtain .9721. Hence log .93775 = 9.9721 − 10.

We now state the following

Rule. — *To find from the table the logarithm of any number*

(1) *Note the position of the decimal point, and write the proper characteristic.*

(2) *Look up in the table the mantissa of the number expressed by the first three significant digits, and that of the next larger number.*

(3) *Find the difference between these mantissas, and multiply that difference by the remaining figures of the given number preceded by a decimal point.*

(4) *Carry this result to the nearest figure in the fourth decimal place, and add it to the mantissa of the first three figures as a correction.*

159. Inverse Use of the Tables. — The inverse use of the table consists in finding the number corresponding to a given logarithm. The number which corresponds to a given logarithm is often called the *antilogarithm.*

Illustrative example I:

Let it be required to find the antilogarithm of 3.7731. The table does not contain characteristics, hence we do not look for the 3. Instead we look among the mantissas for .7731, and find it near the top of the second page of the table. The corresponding number is read by taking the two figures at the left of the page in the same row as the mantissa .7731 for the first two figures, and the figure at the top of the page in the same column as .7731 for the third figure. This gives us 593. Since the characteristic is 3, there are four figures before the decimal point in the number. Hence the desired number is 5930.

The following indicates the procedure when the mantissa cannot be found exactly in the table.

Illustrative example II:

Find the antilogarithm of 1.5744.

We cannot find this mantissa in the table, but we do find the mantissas .5740 and .5752 between which it lies. We write the three figures corresponding to the smaller mantissa .5740, and obtain 375. The difference in the table between .5740 and the mantissa corresponding to 376, .5752, is .0012. The difference between .5740 and the mantissa we are looking for, .5744, is .0004. Now we assume that, since the given mantissa is $\frac{4}{12}$ or $\frac{1}{3}$ of the way between .5740 and .5752, the corresponding number is $\frac{4}{12}$ or $\frac{1}{3}$ of the way between 375 and 376. The work may be arranged as follows:

m 376 corresponds to .5752
m x corresponds to .5744
m 374 corresponds to .5740

$\frac{4}{12} = \frac{1}{3}$

We change $\frac{1}{3}$ to decimal form, carrying it only to the nearest tenth, and annex the result to 375 as the fourth figure of our antilogarithm. This process is also called interpolation. The characteristic 1 shows that there are two figures before the decimal point. Hence the resulting antilogarithm is 37.53.

For finding an antilogarithm, when the given logarithm is not found exactly in the table, we state the following

Rule.—(1) *Locate the given mantissa between two consecutive mantissas in the table, and write the three figures corresponding to the smaller of them.*

(2) *Find the difference between these two consecutive mantissas in the table, and also the difference between the smaller of them and the mantissa whose antilogarithm you desire to find.*

(3) *Divide the latter difference by the former, carrying the result to the nearest tenth.*

(4) *Annex the figure just found to the three figures already written as the fourth figure of the antilogarithm.*

(5) *Insert the decimal point by inspection of the characteristic of the given logarithm.*

Exercises

1. What is the characteristic of the logarithm of each of the following numbers? 2546, 34.87, 5.1256, 46,007, 42,983.6, .8776, 985.17, .00876, .00007.

2. State the position of the decimal point in the numbers whose logarithms have the following characteristics:

3, 5, -1, 6, 0, -2, 2, -3, 4, 9. $\cdots - 10$, 8. $\cdots - 10$, .7. $\cdots - 10$.

Find the logarithm of each of the following numbers:

3. 187.6.	7. .78.	11. 59007.
4. 27658.97.	8. 45000.	12. .00723.
5. 7.435.	9. 3.2876.	13. 653.1.
6. 32.987.	10. 787.77.	14. .00054.

Find the antilogarithm corresponding to each of the following logarithms:

15. 2.9557.	19. 3.8880.	23. .9719.
16. 1.4843.	20. 3.7861.	24. 1.3020.
17. 3.7388.	21. 1.7410.	25. $8.5508 - 10$.
18. .2472.	22. $9.9060 - 10$.	26. $7.4721 - 10$.

160. Operations by Use of Logarithms. — In section 153 you saw how it is possible to multiply two numbers when their exponents to base 2 are known, merely by adding these exponents, and looking up the number corresponding to the resulting exponent. In a similar manner we might divide

by use of that table. For example, suppose it is desired to divide 16,384 by 512. $16{,}384 = 2^{14}$ and $512 = 2^9$. Dividing 2^{14} by 2^9, we obtain 2^5, since we subtract exponents in division. The number which corresponds in the table to 2^5, is 32. Hence $16{,}384 \div 512 = 32$.

In these operations we have really been using logarithms to base 2, but the same principles apply to the use of logarithms to any base.

For the operations of multiplication, division, involution, and evolution we now state the following

Principles. — *The logarithm of a product is equal to the sum of the logarithms, of its factors,* or, stated in symbols,

$$Log\ MN = log\ M + log\ N.$$

Proof: Let $x = \log M$, so that $10^x = M$.
Let $y = \log N$, so that $10^y = N$.
Multiplying these equations, $10^{x+y} = MN$.
Hence, $\log MN = x + y = \log M + \log N$.

The logarithm of a quotient is equal to the logarithm of the dividend or numerator, minus the logarithm of the divisor or denominator, or, stated in symbols,

$$Log\ \frac{M}{N} = log\ M - log\ N.$$

Proof: Let $x = \log M$, so that $10^x = M$.
Let $y = \log N$, so that $10^y = N$.
Dividing the first equation by the second, $10^{x-y} = \frac{M}{N}$.
Hence, $\log \frac{M}{N} = x - y = \log M - \log N$.

The logarithm of a power of a number equals the logarithm of the number multiplied by the exponent of the power, or, stated in symbols,

$$Log\ M^n = n \cdot log\ M.$$

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Proof: Let $x = \log M$, so that $10^x = M$.

Raising both members of this equation to the nth power, $(10^x)^n = 10^{xn} = M^n$.

Hence, $\log M^n = xn = n \cdot \log M$.

The logarithm of a root of a number is the logarithm of the number divided by the index of the root, or, stated in symbols,

$$Log \sqrt[n]{M} = \frac{1}{n} log\ M.$$

Proof: Let $x = \log M$, so that $10^x = M$.

Extracting the nth root of both members of the equation,

$$\sqrt[n]{10^x} = 10^{\frac{x}{n}} = \sqrt[n]{M}.$$

Hence, $\log \sqrt[n]{M} = \frac{x}{n} = \frac{1}{n} \log M$.

For performing arithmetical operations by means of logarithms we now state in harmony with the above principles the following

Rules. — (1) *To multiply two or more numbers add their logarithms and look up the antilogarithm of the result.*

(2) *To divide one number by another subtract the logarithm of the divisor from that of the dividend, and look up the antilogarithm of the result.*

(3) *To raise a number to a power, multiply the logarithm of the number by the exponent and look up the antilogarithm of the result.*

(4) *To extract a root of a number, divide the logarithm of the number by the index of the root, and look up the antilogarithm of the result.*

The operations of addition and subtraction cannot be performed by means of logarithms.

A negative number has no real logarithm to a positive base. Since a negative sign occurring in connection with the operations named will affect only the sign of the result,

problems involving them should be worked as though the signs were all positive. After the numerical value of the result has been obtained, its sign should be inserted according to the usual laws of signs in algebra.

Oral Exercises

1. Using the table on page 362, multiply 4 by 8192, 16 by 32,768, 512 by 512, 64 by 1024, 32 by 16,384.

2. State the characteristic of the logarithm to base 10 of each of the following numbers: 28.16, 147.5, .89, 2164, 4.36, .0172, .0008.

3. State $\log_3 27$, $\log_3 9$, $\log_2 16$, $\log_2 16$, $\log_5 125$, $\log_5 625$, $\log_6 216$, $\log_7 49$, $\log_{25} 5$, $\log_4 2$, $\log_{100} 10$, $\log_8 4$, $\log_{27} 9$.

4. Can 1 be a base of logarithms? Why?

5. What is always the logarithm of the base itself? Why?

6. Show that the logarithm of 1 to any base is 0.

Hint: $a^0 = 1$.

7. The logarithm of 82.96 is 1.9189. What is the logarithm of 8.296? Of 8296? Of .08296?

8. Change $256^2 - 36^2$ so that its value can be computed by logarithms.

Hint: The difference of two squares can be factored.

Illustrative example I:

Compute the value of $\sqrt[3]{181.7} \times 37.486 \times (.24)^2$.

Solution:

Log $\sqrt[3]{181.7} = \frac{1}{3} \log 181.7 = \frac{1}{3}(2.2594)$	=	.7531
Log 37.486	=	1.5738
Log $(.24)^2 = 2 \log .24 = 2(9.3802 - 10)$	=	$18.7604 - 20$
Adding, log answer		$21.0873 - 20$
	=	1.0873
Looking up antilogarithm, answer	=	12.23

Illustrative example II:

Compute the value of $\dfrac{(8.12)^3}{4.687(.016)^2}$.

Solution:

Log $(8.12)^3 = 3 \log 8.12 = 3(.9096)$	$= 2.7288$
Log 4.687	$= .6709$
Log $(.016)^2 = 2 \log .016 = 2(8.2041 - 10)$	$= 16.4082 - 20$
Adding, log of denominator	$= 17.0791 - 20$
Log of numerator may be written	$22.7288 - 20$
Subtracting log of denominator	$17.0791 - 20$
Log of answer	$= 5.6497$
Answer	$= 446400$

161. Exponential Equations. — An exponential equation is an equation in which the unknown quantity occurs in an exponent, as $2^x = 16$ and $3.7^x = 12$. If x has a rational value, it can be determined by inspection. Thus, $x = 4$ in the first example above. In general, however, the solution of exponential equations requires the use of logarithms.

Illustrative example I:

Solve $2^x = 32$ for the value of x without the use of logarithms.

Solution: $2^x = 32.$

$2^x = 2^5.$

$\therefore x = 5.$ Since their bases are the same.

Illustrative example II:

Solve $3^{-x} = 9$ for the value of x, without logarithms.

Solution: $3^{-x} = 9.$

Then $3^x = \frac{1}{9}.$

$3^x = \dfrac{1}{3^2}.$

$3^x = 3^{-2}.$

$\therefore x = -2.$

Illustrative example III:

Express in the language of exponents, $\log_3 9 = x$, and find the value of x.

Solution: Recalling that in your study of logarithms you had to deal with three numbers, (1) a number, (2) its logarithm, (3) the base used, and keeping in mind the fact that $\log_B N = L$, may also be written

$$B^L = N.$$

We then have $3^x = 9.$

Or $3^x = 3^2.$

$$\therefore x = 2.$$

Illustrative example IV:

Solve $3.7^x = 12.$

Solution:

Take the logarithm of both members of the equation.

Then $x \log 3.7 = \log 12.$

$$.5682\, x = 1.0792.$$

$$x = \frac{1.0792}{.5682} = 1.899+.$$

Illustrative example V:

Solve $17^{x-1} = (4.75)^{2x+3}.$

Solution:

$$(x - 1) \log 17 = (2x + 3) \log 4.75.$$

$$(x - 1)\, 1.2304 = (2x + 3)\, .6767.$$

$$1.2304\, x - 1.2304 = 1.3534\, x + 2.0301.$$

$$-.1230\, x = 3.2605.$$

$$x = -26.508.$$

After the numerical value of the logarithm has been inserted in an example like the one just worked, it becomes an ordinary linear equation such as you will find in your first-year algebra. Success requires care in properly placing the decimal point.

Exercises

Find the value of x in each of the following without the use of logarithms:

1. $4^x = 16$.
2. $2^x = 128$.
3. $10^{-x} = \frac{1}{10}$.
4. $2^{x+1} = 32$.
5. $2^{3x+2} = 64$.
6. $2^{2x} = 8$.
7. $(2^2)^{x+1} = 2^{x+5}$.
8. $2^{3x-1} = 2^4 \cdot 2^{x+3}$.
9. $(-2)^{5x} = -32$.
10. $4^{3x-1} = (\frac{1}{2})^{x-1}$.
11. $2^{x^2-1} = 8$.
12. $2^{x^2-4x} = \frac{1}{16}$.
13. Are $(2^x)^x$ and 2^{x^x} the same? Prove, when $x = 3$.

Express in the language of exponents and find the value of each:

14. $\log_3 27 = x$.
15. $\log_4 64 = x$.
16. $\log_4 \frac{1}{16} = x$.
17. $\log_{10} .01 = x$.
18. $\log_8 16 = x$.

Express the following exponential equations in the language of logarithms:

19. (a) $5^2 = 25$. (b) $2^5 = 32$. (c) $7^3 = 343$.

20. What is the logarithm of 9 in a system whose base is 3? Of 81? Of 3? Of $\frac{1}{9}$?

21. Given $\log 2 = .3010$ and $\log 3 = .4771$, find without using the tables the value of $\log 6$, $\log \frac{3}{2}$, $\log 4$, $\log 9$, $\log 12$, $\log \sqrt[2]{2}$, $\log \sqrt[3]{3}$, $\log \sqrt[5]{18}$.

22. Multiply 16.82 by 4.729.

23. Compute $86.43 \times .1861 \times 181.4$.

24. Divide 42.85 by .0137.

25. Compute the value of $\sqrt[5]{81.61}$.

26. Compute $\sqrt[3]{-8.293} \times \sqrt{1086}$.

27. Compute $(2.7)^3 \cdot (38.41)$.

28. Compute $\dfrac{8.1 \times 236.4}{.296}$.

29. Compute $\dfrac{3.471 \times 66.7}{(16.1)^2}$.

30. Compute $(-8.356)^3 \cdot (.012)^2 \cdot \sqrt{677.8}$.

31. Compute $\dfrac{2.875}{(.016)^2\sqrt{29.8}}$.

32. Compute the value of $\dfrac{-2.47 \cdot (3.86)^2 \cdot \sqrt{41.73}}{(-13.61)(2.815)}$.

Solve the equations:

33. $(2.8)^x = 17.72$.

34. $(1.46)^{x-1} = 24.9$.

35. $(278.2)^{x+2} = 16.49$.

36. $(3.75)^x = (4.73)^{x-1}$.

37. $6^x = 5^{2x} \cdot (2.4)^{x-1}$.

38. The volume of a sphere may be computed by the formula $V = \frac{4}{3}\pi r^3$. If $\pi = \frac{22}{7}$, and $r = 6.281$, find V. Compute by logarithms.

39. The volume of a cylinder is found by the formula $V = \pi r^2 h$. If $\pi = \frac{22}{7}$, $r = 4.712$, and $h = 10.96$, compute the value of V by logarithms.

40. A bushel contains 2150.42 cubic inches. Find the number of cubic feet in a bin which will exactly contain 82.76 bushels.

41. Find the number of cords of wood in a pile 32.6 feet long, 12.72 feet wide, and 8.36 feet high. (A cord contains 128 cubic feet.)

42. Find the amount of \$8575 at 4% compound interest, compounded annually, at the end of 10 years if $A = p(1 + r)^n$ where n is the number of years and p is the principal.

43. What annual pension will \$10,000 buy for 20 years if money is worth 4%?

Hint: An. Pension $= Pr \cdot \dfrac{(1 + r)^n}{(1 + r)^n - 1}$

where $P = \$10{,}000$, $n = 20$, $r = .04$.

44. The formula for the amount at the end of n years of $\$p$ placed on interest at $r\%$, and compounded annually is $A = p(1 + r)^n$. In how many years will any amount, say \$100, double itself at 6%, interest compounded annually?

HISTORICAL NOTE

The discovery of the method of logarithms is due to Baron John Napier, a Scotch nobleman, who was born in Merchiston in 1550 and died in 1617. He was deeply interested in political and religious matters and wrote extensively on these topics. He pursued the study of mathematics as a recreation.

Nearly all his discoveries in mathematics were of a kind intended to save labor, and make computations easier, and complicated processes more simple. Logarithms have been called the greatest labor-saving device ever discovered.

We should also remember that while to-day we may think of logarithms in connection with exponential equations, you should also know that exponents were not known to Napier nor were they understood for a century and a half after his time, when Euler noted the relation between exponents and logarithms.

Napier did not use the base 10 which we use to-day. He published a work concerning his discoveries in 1614, and soon thereafter, Henry Briggs, a professor in Oxford University, England, having read this treatise, made a special trip to Scotland to meet Napier.

It is said of this meeting that when Briggs came into the presence of Napier, they greeted each other, and then almost a quarter of an hour elapsed while each looked at the other and neither spoke a word. Finally Briggs spoke as follows: "My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this excellent help in astronomy, viz., logarithms; but, my Lord, being by you found out, I wonder why nobody found it out before, when now known it is so easy."

Doubtless Briggs and Napier talked over the problem of using

John Napier

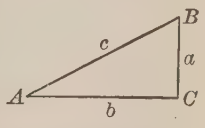
10 as a base, for the Englishman later worked out a set of tables on the base 10. His table extended from 1 to 20,000, and from 90,000 to 100,000, and was calculated to 14 decimal places.

Adrain Vlacq, a Hollander, later (1628) filled in the gap to the table which Briggs had constructed.

Napier's work is probably next in importance after that of Sir Isaac Newton's work on mechanics, the "Principia."

162. Trigonometry is a word derived from the Greek, meaning "triangle-measurement." By means of it we are able to determine the remaining parts of a triangle when three parts are given (except the three angles), and thus solve a variety of practical problems. As the computations of trigonometry are usually, and very conveniently, performed by logarithms, we shall consider the topic briefly in this chapter.

Suppose we have a right triangle as shown in the figure, in which the angles are denoted by the capital letters A, B, and C, and the sides opposite these angles by the corresponding small letters a, b, and c. We shall always mark the vertex of the right angle C and the hypotenuse c.



163. Trigonometric Functions. — The ratio of the side a to the side c, that is $\frac{a}{c}$, in the above figure, is called the sine of the angle A, and is written sin A. Likewise in any other right triangle $A'B'C'$ the sine of the angle A' is $\frac{a'}{c'}$. Now if angle A equals A', the right triangles ABC and $A'B'C'$ are similar. Hence their corresponding sides are proportional, and $\frac{a}{c} = \frac{a'}{c'}$. Hence the ratio $\frac{a}{c}$ has the same value when the angle A is unchanged, regardless of the size of the

triangle. If, however, the angle A is changed, then the value of the ratio $\frac{a}{c}$ is changed. Since the value of the sine depends on the size of the angle, and on nothing else, it is appropriately called a *function* of the angle.

We are now ready to state the following definitions.

The **sine** (abbreviated *sin*) of an acute angle is the ratio of the opposite leg to the hypotenuse.

The **cosine** (abbreviated *cos*) of an acute angle is the ratio of the adjacent leg to the hypotenuse.

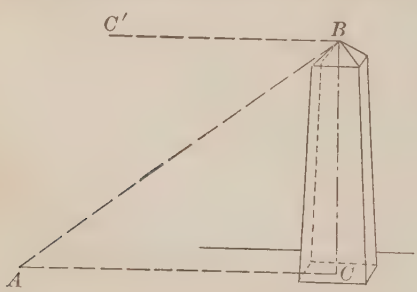
The **tangent** (abbreviated *tan*) of an acute angle is the ratio of the opposite leg to the adjacent leg.

Or in symbols, $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$.

In like manner, $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$, $\tan B = \frac{b}{a}$.

You will notice that when A and B are the acute angles of a right triangle, the sine of A equals the cosine of B, and vice versa.

164. Angles of Elevation and Depression. — In determining the height or depth of objects above or below the horizontal plane, it is necessary to measure angles in a vertical plane. For example, if it were required to find the height of the monument in the diagram, it would be necessary to determine the angle BAC. If, however, we were stationed at B, the angle ABC' would do just as well. Why are these angles equal?



An *angle of elevation or depression* is the angle between the line of sight of the observer and a horizontal line through the point of observation. When this angle is above the horizontal plane it is called an angle of elevation, and when it is below it is called an angle of depression. In the diagram, angle BAC is an angle of elevation, and angle ABC' is an angle of depression.

Both vertical and horizontal angles are measured in practice by an instrument called a *surveyor's transit.*

165. Logarithms of Trigonometric Functions. — The ratios defined in section 163 are pure numbers, and, like all other numbers, they have logarithms to base 10. For example, sin 64° = .8988. Log .8988 is found from our table of logarithms to be 9.9537 − 10. We therefore write log sin 64° = 9.9537 − 10. See page 383.

Tables have been prepared which show the logarithms of the functions of all angles less than 90°. We give here such a table of 4-place logarithms of the sine, cosine, and tangent for every half degree.

166. Direct Use of the Table.

Illustrative example I: Let it be required to find log sin 37.3°. From the table we find log sin 37° = 9.7795 − 10, and log sin 37.5° = 9.7844 − 10. There is a difference between these logarithms of .0049. Since 37.3° is $\frac{3}{5}$ of the distance between 37° and 37.5°, we take $\frac{3}{5}$ of .0049, which equals .0029, and add it to 9.7795 − 10 as a correction, obtaining 9.7824 − 10.

Hence log sin 37.3° = 9.7824 − 10.

Or log sin 37.5° = 9.7844 − 10.

log sin 37° = 9.7795 − 10.

Difference = .0049.

Angle	Log Sin −10 is implied after each log.	Log Cos −10 is implied after each log.	Log Tan −10 is implied after each log.	Angle	Log Sin −10 is implied after each log.	Log Cos −10 is implied after each log.	Log Tan −10 is implied after each log.
0°	—	10.0000	—	23°	9.5919	9.9640	9.6279
0.5°	7.9408	10.0000	7.9409	23.5°	9.6007	9.9624	9.6383
1°	8.2419	9.9999	8.2419	24°	9.6093	9.9607	9.6486
1.5°	8.4179	9.9999	8.4181	24.5°	9.6177	9.9590	9.6587
2°	8.5428	9.9997	8.5431	25°	9.6259	9.9573	9.6687
2.5°	8.6397	9.9996	8.6401	25.5°	9.6340	9.9555	9.6785
3°	8.7188	9.9994	8.7194	26°	9.6418	9.9537	9.6882
3.5°	8.7857	9.9992	8.7865	26.6°	9.6495	9.9518	9.6977
4.	8.8436	9.9989	8.8446	27°	9.6570	9.9499	9.7072
4.5°	8.8946	9.9987	8.8960	27.5°	9.6644	9.9479	9.7165
5°	8.9403	9.9983	8.9420	28°	9.6716	9.9459	9.7257
5.5°	8.9816	9.9980	8.9836	28.5°	9.6787	9.9439	9.7348
6°	9.0192	9.9976	9.0216	29°	9.6856	9.9418	9.7438
6.5°	9.0539	9.9972	9.0567	29.5°	9.6923	9.9397	9.7526
7°	9.0859	9.9968	9.0891	30°	9.6990	9.9375	9.7614
7.5°	9.1157	9.9963	9.1194	30.5°	9.7055	9.9353	9.7701
8°	9.1436	9.9958	9.1478	31°	9.7118	9.9331	9.7788
8.5°	9.1697	9.9952	9.1745	31.5°	9.7181	9.9308	9.7873
9°	9.1943	9.9946	9.1997	32°	9.7242	9.9284	9.7958
9.5°	9.2176	9.9940	9.2236	32.5°	9.7302	9.9260	9.8042
10°	9.2397	9.9934	9.2463	33°	9.7361	9.9236	9.8125
10.5°	9.2606	9.9927	9.2680	33.5°	9.7419	9.9211	9.8208
11°	9.2806	9.9919	9.2887	34°	9.7476	9.9186	9.8290
11.5°	9.2997	9.9912	9.3085	34.5°	9.7531	9.9160	9.8371
12°	9.3179	9.9904	9.3275	35°	9.7586	9.9134	9.8452
12.5°	9.3353	9.9896	9.3458	35.5°	9.7640	9.9107	9.8533
13°	9.3521	9.9887	9.3634	36°	9.7692	9.9080	9.8613
13.5°	9.3682	9.9878	9.3804	36.5°	9.7744	9.9052	9.8692
14°	9.3837	9.9869	9.3968	37°	9.7795	9.9023	9.8771
14.5°	9.3986	9.9859	9.4127	37.5°	9.7844	9.8995	9.8850
15°	9.4130	9.9849	9.4281	38°	9.7893	9.8965	9.8928
15.5°	9.4269	9.9839	9.4430	38.5°	9.7941	9.8935	9.9006
16°	9.4403	9.9828	9.4575	39°	9.7989	9.8905	9.9084
16.5°	9.4533	9.9817	9.4716	39.5°	9.8035	9.8874	9.9161
17°	9.4659	9.9806	9.4853	40°	9.8081	9.8843	9.9238
17.5°	9.4781	9.9794	9.4987	40.5°	9.8125	9.8810	9.9315
18°	9.4900	9.9782	9.5118	41°	9.8169	9.8778	9.9392
18.5°	9.5015	9.9770	9.5245	41.5°	9.8213	9.8745	9.9468
19°	9.5126	9.9757	9.5370	42°	9.8255	9.8711	9.9544
19.5°	9.5235	9.9743	9.5491	42.5°	9.8297	9.8676	9.9621
20°	9.5341	9.9730	9.5611	43°	9.8338	9.8641	9.9697
20.5°	9.5443	9.9716	9.5727	43.5°	9.8378	9.8606	9.9772
21°	9.5543	9.9702	9.5842	44°	9.8418	9.8569	9.9848
21.5°	9.5641	9.9687	9.5954	44.5°	9.8457	9.8532	9.9924
22°	9.5736	9.9672	9.6064	45°	9.8495	9.8495	10.0000
22.5°	9.5828	9.9656	9.6172				

Angle	Log Sin −10 is implied after each log.	Log Cos −10 is implied after each log.	Log Tan	Angle	Log Sin −10 is implied after each log.	Log Cos −10 is implied after each log.	Log Tan
45°	9.8495	9.8495	.0000	68°	9.9672	9.5736	.3936
45.5°	9.8532	9.8457	.0076	68.5°	9.9687	9.5641	.4046
46°	9.8569	9.8418	.0152	69°	9.9702	9.5543	.4158
46.5°	9.8606	9.8378	.0228	69.5°	9.9716	9.5443	.4273
47°	9.8641	9.8338	.0303	70°	9.9730	9.5341	.4389
47.5°	9.8676	9.8297	.0379	70.5°	9.9743	9.5235	.4509
48°	9.8711	9.8255	.0456	71°	9.9757	9.5126	.4630
48.5°	9.8745	9.8213	.0532	71.5°	9.9770	9.5015	.4755
49°	9.8778	9.8169	.0608	72°	9.9782	9.4900	.4882
49.5°	9.8810	9.8125	.0675	72.5°	9.9794	9.4781	.5013
50°	9.8843	9.8081	.0762	73°	9.9806	9.4659	.5147
50.5°	9.8874	9.8035	.0839	73.5°	9.9817	9.4533	.5284
51°	9.8905	9.7989	.0916	74°	9.9828	9.4403	.5425
51.5°	9.8935	9.7941	.0994	74.5°	9.9839	9.4269	.5570
52°	9.8965	9.7893	.1072	75°	9.9849	9.4130	.5719
52.5°	9.8995	9.7844	.1150	75.5°	9.9859	9.3986	.5873
53°	9.9023	9.7795	.1229	76°	9.9869	9.3837	.6032
53.5°	9.9052	9.7744	.1308	76.5°	9.9878	9.3682	.6196
54°	9.9080	9.7692	.1387	77°	9.9887	9.3521	.6366
54.5°	9.9107	9.7640	.1467	77.5°	9.9896	9.3353	.6542
55°	9.9134	9.7586	.1548	78°	9.9904	9.3179	.6725
55.5°	9.9160	9.7531	.1629	78.5°	9.9912	9.2997	.6915
56°	9.9186	9.7476	.1710	79°	9.9919	9.2806	.7113
56.5°	9.9211	9.7419	.1792	79.5°	9.9927	9.2606	.7320
57°	9.9236	9.7361	.1875	80°	9.9934	9.2397	.7537
57.5°	9.9260	9.7302	.1958	80.5°	9.9940	9.2176	.7764
58°	9.9284	9.7242	.2042	81°	9.9946	9.1943	.8003
58.5°	9.9308	9.7181	.2127	81.5°	9.9952	9.1697	.8255
59°	9.9331	9.7118	.2212	82°	9.9958	9.1436	.8522
59.5°	9.9353	9.7055	.2299	82.5°	9.9963	9.1157	.8806
60°	9.9375	9.6990	.2386	83°	9.9968	9.0859	.9109
60.5°	9.9397	9.6923	.2474	83.5°	9.9972	9.0539	.9433
61°	9.9418	9.6856	.2562	84°	9.9976	9.0192	.9784
61.5°	9.9439	9.6781	.2652	84.5°	9.9980	8.9816	1.0164
62°	9.9459	9.6716	.2743	85°	9.9983	8.9403	1.0580
62.5°	9.9479	9.6644	.2835	85.5°	9.9987	8.8946	1.1040
63°	9.9499	9.6570	.2928	86°	9.9989	8.8436	1.1554
63.5°	9.9518	9.6495	.3023	86.5°	9.9992	8.7857	1.2135
64°	9.9537	9.6418	.3118	87°	9.9994	8.7188	1.2806
64.5°	9.9555	9.6340	.3215	87.5°	9.9996	8.6397	1.3599
65°	9.9573	9.6259	.3313	88°	9.9997	8.5428	1.4569
65.5°	9.9590	9.6177	.3413	88.5°	9.9999	8.4179	1.5819
66°	9.9607	9.6093	.3514	89°	9.9999	8.2419	1.7581
66.5°	9.9624	9.6007	.3617	89.5°	10.0000	7.9408	2.0591
67°	9.9640	9.5919	.3721	90°	10.0000	—	
67.5°	9.9656	9.5828	.3828				

Illustrative example II:

Find log tan 63.2°.

63.2° is $\frac{2}{5}$ of the distance between 63° and 63.5°.

Log tan 63°	= .2928.
Log tan 63.5°	= .3023.
Difference	= .0095.
$\frac{2}{5}$(.0095)	= .0038.

Hence log tan 63.2° = .2928 + .0038 = .2966.

Illustrative example III:

Find log cos 40.8°.

40.8° is $\frac{3}{5}$ of the way between 40.5° and 41°.

Log cos 40.5°	= 9.8810 − 10.
Log cos 41°	= 9.8778 − 10.
Difference	= .0032.
$\frac{3}{5}$(.0032)	= .0019.

Hence log cos 40.8° = (9.8810 − 10) − .0019 = 9.8791 − 10.

Notice that the cosine is a *decreasing* function, that is, cos 41° is less than cos 40.5°. Hence the correction is *subtracted* instead of being *added*, as was done in the cases of the sine and tangent.

These examples illustrate the following

Rule. — (1) *If the given angle is not found exactly in the table, locate it between two successive angles which are given, and determine its proportionate position between them.*

(2) *Find the difference between the logarithms of these successive angles in the table, and multiply it by the fraction which expresses the proportionate location of the given angle between them.*

Add to the logarithm corresponding to the smaller angle in case log sin or log tan is desired, and subtract it in case log cos is desired.

167. Inverse Use of the Table.

Illustrative example I:

Find the angle X, if $\log \sin X = 9.9298 - 10$.

We locate the given logarithm in the table between $\log \sin 58° = 9.9284 - 10$ and $\log \sin 58.5° = 9.9308 - 10$. The difference between these two logarithms is .0024, and this corresponds to a difference of .5° in the angle. The difference between the logarithm corresponding to the smaller of these angles and the given logarithm is .0014. Hence the required angle $= \frac{.0014}{.0024} = \frac{7}{12}$ of the way between 58° and 58.5°. $\frac{7}{12} \cdot \frac{1}{2}° = \frac{7}{24}° = .3°$.

Hence $X = 58.3°$.

Illustrative example II:

Find the angle X if $\log \cos X = 9.6840 - 10$.

We locate the given logarithm between $\log \cos 61° = 9.6856 - 10$ and $\log \cos 61.5 = 9.6787 - 10$.

The difference between $9.6856 - 10$ and $9.6787 - 10$ is $-.0069$.

The difference between $9.6856 - 10$ and $9.6840 - 10$ is $-.0016$.

Hence the required angle lies $\frac{-.0016}{-.0069} = \frac{16}{69}$ of the way between 61° and 61.5°. $\frac{16}{69} \cdot \frac{1}{2}° = \frac{8}{69}° = .1°$.

Therefore $X = 61.1°$.

Illustrative example III:

Find the angle X, if $\log \tan X = .0186$.

We locate the given logarithm between $\log \tan 46° = .0152$ and $\log \tan 46.5° = .0076$.

The difference between $\log \tan 46°$ and $\log \tan 46.5° = .0076$.

The difference between .0152 and .0186 = .0034.

$$\frac{.0034}{.0076} = \frac{34}{76} = \frac{17}{38}. \quad \frac{17}{38} \cdot \frac{1^\circ}{2} = \frac{17^\circ}{76} = .2^\circ.$$

Hence $X = 46.2^\circ$.

Exercises

Find:

1. log sin 82.3°.
2. log sin 17.8°.
3. log sin 63.4°.
4. log cos 28.3°.
5. log cos 48.7°.
6. log cos 37.1°.
7. log tan 6.1°.
8. log tan 82.6°.
9. log tan 49.4°.

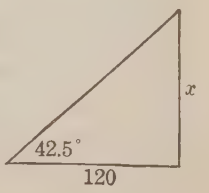
Find the angle X if

10. log tan $X = 8.8690 - 10$
11. log tan $X = 9.2307 - 10$.
12. log tan $X = .3168$.
13. log sin $X = 9.7325 - 10$.
14. log sin $X = 9.2162 - 10$.
15. log sin $X = 9.3340 - 10$.
16. log cos $X = 9.0076 - 10$.
17. log cos $X = 9.7743 - 10$.

168. The Solution of Problems. — In solving problems by the use of these tables you should draw a figure as in the illustrative examples which follow.

Illustrative example I:

A boy standing on level ground notices that the angle of elevation to the top of a flagpole is 42.5°. He measures the distance to the foot of the pole and finds it to be 120 feet. How high is the pole?



Solution: $\frac{x}{120} = \tan 42.5°.$

$$x = 120 \tan 42.5°.$$

Hence $\log x = \log 120 + \log \tan 42.5°.$

$$\log 120 = 2.0792$$
$$\log \tan 42.5° = 9.9621 - 10$$
$$\log x = 2.0413$$
$$x = 110 \text{ feet.}$$

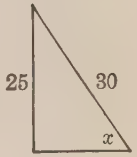
Illustrative example II:

A ladder 30 feet long just reaches a window 25 feet from the ground. What angle does the ladder make with the ground?

Solution: $\sin X = \frac{25}{30}.$

Hence $\log \sin X = \log 25 - \log 30.$

$$\log 25 = 1.3979$$
$$\log 30 = 1.4771$$
$$\log \sin X = 9.9208 - 10$$
$$X = 56.4°.$$



Illustrative example III:

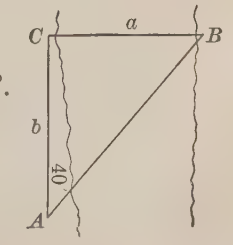
A man standing on the bank of a river at C wishes to find the distance to the point B on the opposite side. He locates the point A so that ABC forms a right triangle. By measuring he finds that the distance AC is 250 feet, and the angle at A he finds by use of the transit to be 40°. Find the distance BC.

Solution: $\frac{a}{250} = \tan 40°.$

Hence $a = 250 \cdot \tan 40°$

and $\log a = \log 250 + \log \tan 40°.$

$$\log 250 = 2.3979$$
$$\log \tan 40° = 9.9238 - 10$$
$$\log a = 2.3217$$
$$a = 209.8.$$



It is a good thing to check the solution of problems in trigonometry by drawing a figure to scale and measuring the length of the unknown line, or the size of the unknown angle by means of a protractor. Squared paper may be used to advantage in such constructions.

Exercises

1. In a right triangle the hypotenuse is 15 inches and the smallest angle is 22°. How long is the side opposite that angle? The side adjacent to that angle?

2. An airplane is directly over a town which is known to be 10 miles distant from the observer. The angle of elevation of the airplane is found to be 15°. How high is the airplane?

3. The angle of elevation of a tower is 30°, viewed from a point 300 yards distant on level ground. How high is the tower?

4. The angle to the top of a cliff at the side of a river, as viewed from the opposite side, is 16.5°. How high is the cliff if the river is 450 feet wide?

5. The length of a kite string is 220 yards and the angle of elevation of the kite is 42°. Find the height of the kite if the kite string is straight.

6. Find the angle A if there is a rise of 1 foot in 10 feet of railroad track running approximately in a straight line.

7. A wedge is in the shape of a right triangle. How wide is the wedge if its length measured on the hypotenuse is 20 inches and the angle at its edge is 10°?

8. If a right triangle whose legs are 5 inches and 12 inches is constructed, the greatest acute angle is often taken as $\frac{3}{4}$ of a right angle. How nearly correct is this?

9. What is the angle of elevation of the sun if a pole casts a shadow whose length is the same as the length of the

pole? Check with graph paper, taking the length of the pole as 75 feet.

10. If a lighthouse is 300 feet above the sea level and the angle of elevation of the light as observed from a ship is 10°, how far from the ship is the lighthouse?

11. If a certain railroad track rises 1198 feet for 1 mile of track, what is its angle of elevation?

12. A dam 24 feet high forces the water in a stream back a horizontal distance of $\frac{1}{4}$ mile. What is the slope of the ground?

13. When an airplane is 1000 feet high, the angle of depression to a building is 72°. How far is the building from the airplane?

14. The distance AC on level ground along a stream is 65 feet. The angle A is measured and found to be 50.7°. The triangle ABC is a right triangle. Find the distance across the stream.

15. The distance from a point P on level ground to the foot of a building is 300 feet. The angle of elevation of the top of the building from P is 60°. What is the height of the building? What is the length of a straight wire from P to the top of the building?

16. If the angle of depression of the river flowing through the Grand Cañon of the Colorado when observed from the brink of the cañon at a horizontal distance of 3 miles is 17.6°, what is the depth of the cañon?

169. Areas of Triangles. — It can be proved that the area of any triangle equals half the product of any two sides by the sine of the included angle. Hence in formula form

$$\text{Area} = \tfrac{1}{2}\, ab \sin C.$$

Therefore, $\log \text{area} = \log a + \log b + \log \sin C - \log 2.$

Exercises

1. If side $a = 34.56$ feet, side $b = 97.45$ feet, and the angle $C = 56.4°$, find the area.

2. If angle $A = 34.8°$, $b = 11.78$, and $c = 21.7$, find the area.

3. Two of the sides of a military camp were 4 miles and 6 miles, and the angle between them 35.6°. Find the area of the camp.

4. A lot in the form of a triangle has two sides of 25 feet and 37 feet and the angle between them is 34.5°. Find the area of the triangle.

HISTORICAL NOTE

Trigonometry.

Hipparchus, a Greek, who lived about 150 B.C., was the first to make any progress in this subject. He was able to calculate the length of a year to within six minutes.

Some advances were made by the Hindus (500 A.D.) and the Arabs (800 A.D. to 1000 A.D.), but it was Regiomontanus, a German (1436–1476), who made the science somewhat independent of astronomical calculations.

It was in the seventeenth century, however, that the greatest advance was made, owing to the invention of logarithms and the fact that algebraic symbolism became much more nearly perfected.

CHAPTER XVII

SUPPLEMENTARY TOPICS

THE body of the book contains ample material for either a half-year or a year course, following first-year algebra. It has been thought best, however, to present in the present chapter a few supplementary topics for the use of such schools as may desire them.

170. The Remainder Theorem may be stated thus:

If a polynomial in x is divided by $x - c$, the remainder is the result obtained by substituting c in the place of x in the polynomial.

Proof: Let the polynomial be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a^n.$$

Divide the polynomial by $x - c$, and let Q denote the quotient and R the remainder.

Then $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a^n \equiv (x - c)Q + R.$

In this identity let $x = c.$

Then $a_0c^n + a_1c^{n-1} + a_2c^{n-2} + \cdots a^n \equiv O \cdot Q + R \equiv R.$

That is, the remainder is equal to the original polynomial with c substituted for x.

Illustrative example:

Find the remainder when $2x^3 - 6x^2 + 5x - 2$ is divided by $x - 3$.

Solution: Substituting 3 for x in the polynomial, we obtain

$$2 \cdot 27 - 6 \cdot 9 + 5 \cdot 3 - 2 = 54 - 54 + 15 - 2 = 13.$$

Hence the remainder is 13. Verify by division.

How would you find the remainder when the above polynomial is divided by $x + 2$? *Hint:* $x + 2 = x - (-2)$.

171. The Factor Theorem, which has already been stated and proved in section 40, is a special case of the Remainder Theorem. It states that if the result of substituting c for x in a polynomial is 0, then $x - c$ is a factor of that polynomial. This becomes apparent at once, now that we know that the result of such substitution represents the remainder when the polynomial is divided by $x - c$.

Exercises

1. Find the remainder when $x^4 - 3x^2 + 2x - 1$ is divided by $x - 2$; by $x - 1$; by $x + 3$.

2. Find the remainder when $2x^4 - 4x^3 + x^2 - 4x + 4$ is divided by $x - 1$; $x + 1$; $x - 2$. Which of these is a factor of the polynomial? Why?

3. Find the remainder when $x^5 + 32$ is divided by $x - 2$.

4. Show that $x - 1$ is a factor of $x^3 - 4x + 3$. What is the other factor?

5. Find all the factors of $x^3 - 2x^2 - 5x + 6$.

6. Can $2x^4 - x^3 + 6x^2 + 3$ be divided evenly by $x - 3$?

172. Division by Detached Coefficients. — The labor of dividing one polynomial by another may be somewhat shortened by a method called *division* by *detached coefficients* in which only the coefficients of the terms of the dividend and divisor are written. We shall illustrate this method by dividing $3x^4 + x^3 - 2x^2 + 5x - 3$ by $3x^2 - 2x + 3$.

Solution by detached coefficients.

$$
\begin{array}{r|l}
3 + 1 - 2 + 5 - 3 & 3 - 2 + 3 \\
\underline{3 - 2 + 3} & \boxed{1 + 1 - 1} \\
\quad 3 - 5 + 5 & \\
\quad \underline{3 - 2 + 3} & \text{Hence the quotient is } x^2 + x - 1. \\
\quad\quad - 3 + 2 - 3 & \\
\quad\quad \underline{- 3 + 2 - 3} &
\end{array}
$$

Solution with the usual process.

$$\begin{array}{l|l} 3x^4 + x^3 - 2x^2 + 5x - 3 & 3x^4 - 2x^3 + 3x^2 \\ \underline{3x^4 - 2x^3 + 3x^2} & \underline{x^2 + x - 1} \\ 3x^3 - 5x^2 + 5x & \\ \underline{3x^3 - 2x^2 + 3x} & \\ -3x^2 + 2x - 3 & \\ \underline{-3x^2 + 2x - 3} & \end{array}$$

In dividing by this method both dividend and divisor must be arranged in the same order and coefficients of missing powers must be supplied by zeros.

173. Synthetic Division. — An especially useful form of division by detached coefficients is that which is called Horner's Method of synthetic division.

This method is of more value when the divisor is of the form $x \pm a$, and we shall confine our discussion to this case. The following example illustrates the method.

Illustrative example I : Divide

$$x^3 + 4x^2 + 3x - 1 \text{ by } x - 3.$$

Solution:

$$\begin{array}{rrrrl} 1 & 4 & 3 & -1 & \lfloor 3 \\ & 3 & 21 & 72 & \\ \hline 1 & 7 & 24 & 71 & \end{array}$$

The quotient is $x^2 + 7x + 24$ with a remainder of 71.

Explanation: The terms of the dividend are written down in order, using only detached coefficients. In the divisor the x term is dropped and the second term is written with opposite sign. We then draw a line under the dividend, leaving room for a row of figures above it. We then bring down the first term 1. Then this 1 is multiplied by the 3 in the divisor, and the resulting 3 is written under the 4. Add. The resulting 7 is multiplied by the 3 and the product written

under the 3 of the dividend. Again add. We continue by the same process to the end of the line. The last sum (71) is the remainder and the other numbers are the coefficients of the quotient in order.

Illustrative example II: Divide $x^4 + 6x^2 + 2x - 3$ by $x + 1$.

Solution:

$$\begin{array}{rrrrrr|l} 1 & 0 & 6 & 2 & -3 & & -1 \\ & -1 & 1 & -7 & 5 & & \\ \hline 1 & -1 & 7 & -5 & 2 & & \end{array}$$

The quotient is $x^3 - x^2 + 7x - 5$ and the remainder is 2. Note the necessity of writing as 0 the coefficient of the x^3 term which is lacking in the dividend.

Exercises

Using detached coefficients, divide

1. $4x^4 + 3x^2 + 4x - 3$ by $2x^2 - x + 3$.
2. $2x^4 - 13x^3 - 8x^2 + 22x - 3$ by $x^2 - 7x + 1$.

Using synthetic division, divide

3. $x^3 - 4x^2 + 2x - 2$ by $x + 1$; by $x - 2$; by $x - 3$.
4. $x^4 + 3x^3 + x - 2$ by $x + 2$; by $x - 4$.
5. $2x^5 + 3x^3 + 4x^2 - 1$ by $x + 1$.
6. $x^5 - 32$ by $x + 2$.
7. $2x^3 + 3x^2 - x + 1$ by $x - 2$.
8. Find the remainder when $2x^3 + 3x^2 - x + 1$ is divided by $x - 2$, using the remainder theorem. Compare with the result of example 7.
9. Suppose that it is desired to graph the function $y = 2x^3 + 3x^2 - x + 1$. Instead of substituting the values -2, -1, 0, 1, 2, 3, etc., for x to obtain the corresponding values of y, explain how we may obtain these same results

by synthetic division. Graph the above function, obtaining your tables of values by synthetic division.

10. Divide $x^3 - 10x + 3$ by $x - 3$, using synthetic division. What is the remainder? State a factor of $x^3 - 10x + 3$. State a root of the equation $x^3 - 10x + 3 = 0$.

See if you can use the method of detached coefficients in the opposite operation to that of division, namely, multiplication, in each of the following:

11. $(2x + 4)(3x + 5)$.

12. $(x^2 + 3x + 4)(2x + 3)$.

13. $(3x^4 + 5x^3 + 2x^2 + 6x + 1)(2x^2 + 3x + 4)$.

14. $(x^2 + 3x + 6)(2x^3 - 5x + 4)$.

15. $(3x^4 - 5x + 2x^3 + 4)(4x^3 + 2x^2 + x + 2)$.

174. Mathematical Induction proves a thing by proceeding from the particular to the general. This was illustrated in our proof shown in section 69. It would also be illustrated in the complete proof of the Binomial Theorem of section 150. It may be further shown by a few examples.

Illustrative example I:

Prove by mathematical induction that the sum of the first n natural numbers is $\frac{n}{2}(n + 1)$.

Solution:

$$1 + 2 = 3 = \tfrac{1}{2} \cdot 2(2 + 1)$$
$$1 + 2 + 3 = 6 = \tfrac{1}{2} \cdot 3(3 + 1)$$
$$1 + 2 + 3 + 4 = 10 = \tfrac{1}{2} \cdot 4(4 + 1)$$
$$1 + 2 + 3 + 4 + 5 = 15 = \tfrac{1}{2} \cdot 5(5 + 1)$$
$$1 + 2 + 3 + 4 + 5 + 6 = 21 = \tfrac{1}{2} \cdot 6(6 + 1)$$

Thus far the formula seems to hold good. In other words the sum of the first n whole numbers appears to conform with the above law. Now let us try the next consecutive integer, viz., $n + 1$. We obtain

$$1+2+3+4+5+\cdots+n+(n+1) = \frac{n+1}{2}(1+n+1)$$
$$= \tfrac{1}{2}(n+1)(n+2)$$
$$= \tfrac{1}{2}(n+1)[(n+1)+1].$$

This is of the same form as that of the law, therefore we conclude that the law not only holds good for n terms but also for $(n+1)$ terms, and we then conclude that the law holds for any number of terms. This gives us the following

Rule *for proving by mathematical induction:*

(1) *Prove that the law holds good when* $n = 1$, *or* 2, *or* 3, *etc.*

(2) *Prove that if the law holds good for any particular value of* n, *as in* (1), *then it also holds good when* $(n+1)$ *is substituted for* n.

(3) *We now conclude that the law holds good for any value whatsoever that we may assign to* n.

Illustrative example II:

We wish to prove by mathematical induction that the sum of n terms of the sequence

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof: If $n = 2$, $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3} = \dfrac{1}{2}+\dfrac{1}{6} = \dfrac{2}{3}$,

and $$\frac{n}{n+1} = \frac{2}{2+12} = \frac{2}{3}.$$

Now let us assume that it does hold when we use n terms and try to prove it when we replace n with $(n+1)$.

We shall do this by adding the sum of n terms, viz., $\dfrac{n}{n+1}$ to the next term, and we have $\dfrac{n}{n+1}+\dfrac{1}{(n+1)(n+2)}$. By addition we actually find that this equals $\dfrac{n+1}{n+2}$.

Hence we observe that if the sum of n terms is $\frac{n}{n+1}$ then it follows that the sum of $(n+1)$ terms is $\frac{n+1}{n+2}$.

In the third step we found that the sum of 2 terms was $\frac{2}{3}$. If we were to find the sum of 3 terms, it would be $\frac{3}{4}$.

Thus we observe that $\frac{n}{n+1}$ holds good for any sequence.

Illustrative example III:

Prove by mathematical induction that $a^n - b^n$ is always exactly divisible by $a - b$.

Proof:

$$\begin{array}{r|l} a^n - b^n & a - b \\ \hline a^n - a^{n-1}b & a^{n-1} \\ \hline a^{n-1}b - b^n = b(a^{n-1} - b^{n-1}) & \end{array}$$

Now if $a^{n-1} - b^{n-1}$ is exactly divisible by $a - b$, then our division will come out without any remainder.

Thus, if $a^{n-1} - b^{n-1}$ (the difference of the same powers of a and b) is exactly divisible by $a - b$, then the difference of the next higher power of $a^n - b^n$ is exactly divisible by $a - b$.

Hence if $a^2 - b^2$ is exactly divisible by $a - b$, then $a^3 - b^3$ is also exactly divisible by $a - b$, and so on indefinitely.

Illustrative example IV:

Prove by mathematical induction that

$$a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$$

Proof: By trial this formula for the sum of n terms is found to be true when $n = 2$, and when $n = 3$.

Let n have a value for which it is known to be true as $n = 3$, then for this value of n

$$a + (a + d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$$

is true.

Add $a + nd$ to both members of the equation, thus giving the series $(n + 1)$ terms.

Then $a + (a + d) + \cdots + [a + (n - 1)d] + (a + nd)$

$$= \frac{n}{2}[2a + (n - 1)d] + (a + nd)$$

$$= an + \frac{dn^2}{2} - \frac{dn}{2} + a + nd$$

$$= \frac{2an + dn^2 + dn + 2a}{2}$$

$$= \frac{dn(n + 1) + 2a(n + 1)}{2}$$

$$= \frac{n + 1}{2}(2a + nd)$$

$$= \frac{n + 1}{2}(2a + [(n - 1) + 1]d).$$

This expression for $(n + 1)$ terms has the same *form* with reference to $(n + 1)$ that the original formula for the sum of n terms has with reference to n. Then if the formula is true for any number of terms it is true for one more term. But it is true for $n = 3$, hence it is true for $n = 4$, therefore true for $n = 5$, and so on for all values of n.

Exercises

Prove by mathematical induction that

1. The sum of the first n even numbers is $n(n + 1)$.
2. The sum of the first n odd numbers equals n^2.
3. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$.
4. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n}{3}(n + 1)(2n + 1)$.
5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n}{2}(n + 1)\right]^2$.

6. $1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1} = \dfrac{3^n - 1}{2}$.

7. $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

8. $3 + 6 + 9 + \cdots + 3n = \dfrac{3}{2} n(n + 1)$.

9. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n}{3} (n+1)(n+2)$.

10. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$.

11. $x^{2n} - y^{2n}$ is exactly divisible by $x + y$ when n is a positive integer.

Hint: See illustrative example III and carry division until two terms are obtained in the quotient.

12. $x^n + y^n$ is exactly divisible by $x + y$ when n is odd.

13. $2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$.

14. $3^{2n} - 1$ is exactly divisible by 8.

15. Illustrative example III by the remainder theorem.

16. The sum of $1^3 + 3^3 + 5^3 + \cdots$ to n terms is $n^2(2n^2 - 1)$.

175. Harmonic Progression is defined as a series of terms whose reciprocals are in arithmetic progression. For example,

$$1, \tfrac{4}{3}, 2, 4, \cdots$$

is a harmonic progression since the reciprocal of these terms

$$1, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{1}{4}, \cdots$$

constitute an arithmetic progression.

176. Harmonic Mean. — A formula for the harmonic mean between a and b can be derived thus:

Let us represent the required mean by M.

Then a, M, b form a harmonic progression.

Hence $\frac{1}{a}, \frac{1}{M}, \frac{1}{b}$ form an arithmetic progression.

Hence
$$\frac{1}{M} - \frac{1}{a} = \frac{1}{b} - \frac{1}{M}.$$
$$\frac{2}{M} = \frac{1}{a} + \frac{1}{b}.$$
$$2ab = aM + bM.$$
$$M = \frac{2ab}{a+b}.$$

If more than one harmonic mean is desired to be inserted between two terms, *invert the given terms, insert the required number of arithmetic means, and invert the terms of the result.*

Illustrative example: Insert 5 harmonic means between $\frac{1}{2}$ and $\frac{1}{5}$.

Solution: We insert 5 arithmetic means between 2 and 5.

Thus
$$l = a + (n-1)d.$$
$$5 = 2 + 6d.$$
$$6d = 3.$$
$$d = \tfrac{1}{2}.$$

The arithmetic series then is $2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5$.

The harmonic series with terms inverted is

$$\tfrac{1}{2}, \tfrac{2}{5}, \tfrac{1}{3}, \tfrac{2}{7}, \tfrac{1}{4}, \tfrac{2}{9}, \tfrac{1}{5}$$

In general *to solve a problem in harmonic progression, invert the given quantities. Solve the corresponding problem in Arithmetic Progression. Invert the results.*

Exercises

In the harmonic series

1. Find the 8th term of $8, \frac{16}{3}, 4, \cdots$
2. Find the 6th term of $2 + \sqrt{3}, \frac{1}{2}, 2 - \sqrt{3}, \cdots$
3. Find the 10th term of $\frac{1}{3}, \frac{3}{10}, \frac{3}{11}, \cdots$

4. The third term of a harmonic progression is $\frac{1}{5}$, and the 6th term is $-\frac{1}{4}$. Find the 10th term.

5. Insert 3 harmonic means between $-\frac{1}{2}$ and $\frac{1}{14}$.

6. Insert 4 harmonic means between 8 and -12.

7. Insert 3 harmonic means between h and k.

8. The first two terms of a harmonic progression are 5 and 6. Which term of the series will be 30?

9. If A and H represent the arithmetic mean and the harmonic mean between x and y respectively, prove that the geometric mean between x and y is also the geometric mean between A and H.

10. If vibrating strings have the same diameter and tension, their pitch depends solely on their length. Strings which produce the major chord do, mi, sol, under these conditions have the relative lengths 1, $\frac{4}{5}$, $\frac{2}{3}$. Show that these numbers are in harmonic progression.

177. Other Simple Series are illustrated under progressions in Chapter XIV, and formulas for the sum of a few types are found under mathematical induction.

178. Cube Roots of Polynomials. — The plan may be followed by a study of the formula

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

Illustrative example I:

Find the cube root of $a^3 + 3\,a^2b + 3\,ab^2 + b^3$.

Solution: Observe that the first term of the result is the cube root of a^3 or a. The second term of the cube root is obtained by squaring a, multiplying it by 3, and dividing this result as a trial divisor into the second term of the given polynomial, obtaining in this example b. The arrangement of the work is as follows.

$a^3+3\,a^2b+3\,ab^2+b^3 \mid a+b$

a^3

Trial divisor $3,\ a^2 = 3\,a^2 \mid 3\,a^2b+3\,ab^2+b^3$

Second term of the root

$3\,a^2b \div 3\,a^2 = b.$

Complete divisor $3\,a^2+3\,ab+b^2$

$b(3\,a^2+3\,ab+b^2) = 3\,a^2b+3\,ab^2+b^3$

Illustrative example II:

Find the cube root of $27\,a^3 + 36\,ab^2 + 54\,a^2b + 8\,b^3$.

Solution:

$27\,a^3 + 54\,a^2b + 36\,ab^2 + 8\,b^3 \mid 3\,a + 2\,b$

$27\,a^3$

$54\,a^2b + 36\,ab^2 + 8\,b^3 \mid 3(3\,a)^2 = 27\,a^2$

$54\,a^2b + 36\,ab^2 + 8\,b^3 \mid (27\,a^2 + 18\,ab + 4\,b^2)$

Explanation: The example should first be arranged according to the ascending or descending powers of some letter.

The cube root of $27\,a^3$ is $3\,a$, which is the first term of the root.

Trial divisor is $3(3\,a)^2 = 27\,a^2$.

$54\,a^2b \div 27\,a^2 = 2\,b$, the second term of the root.

Complete divisor is $27\,a^2 + 18\,ab + 4\,b^2$.

This is then multiplied by the second term of the root, etc.

Exercises

Find the cube root of each of the following expressions:

1. $a^3 + 9\,a^2 + 27\,a + 27.$
2. $a^3 + 6\,a^2b + 12\,ab^2 + 8\,b^3.$
3. $8\,x^3 - 36\,x^2y + 54\,xy^2 - 27\,y^3.$
4. $x^3 + 12\,x^2 + 48\,x + 64.$

5. $b^6 - 3ab^5 + 5a^3b^3 - 3a^5b - a^6$.

6. $a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1$.

7. $a^6 - \frac{3a^4}{2} + \frac{3a^2}{4} - \frac{1}{8}$.

8. $x^6 + 6x^4 + 3x^5 + 6x^2 + 7x^3 + 3x + 1$.

9. $8y^3 - 12y^2 + 12y - 7 + \frac{3}{y} - \frac{3}{4y^2} + \frac{1}{8y^3}$.

10. $\frac{1}{8} + \frac{3}{4}x + \frac{3}{2}x^2 + x^3$.

11. Formulate a rule for finding cube root of polynomials.

Find the sixth root of the following by finding the square root, and then finding the cube root of the result.

12. $64x^{12} - 192x^{10} + 240x^8 - 160x^6 + 60x^4 - 12x^2 + 1$.

179. Cube Root of Arithmetical Numbers. — The method of performing this work is very similar to that explained in the preceding section. It is also helpful to recall that the cube root of any number between 1 and 1000 lies between 1 and 10; the cube root of any number between 1000 and 1,000,000 lies between 10 and 100, and so on. Cube root of numbers, however, is not obtained very much by this method in actual practice but by means of logarithms or by a table similar to the one which you will find on page 455.

We shall ask you to think of any number, whose cube root is to be obtained, as in the form of

$$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3,$$

where the t stands for the number of tens and the u represents the number of units. Then from the right-hand member of the above equation we find that the cube of every number which has both tens and units digits may be stated: *the cube of the tens plus three times the square of the tens times the units plus three times the tens times the square of the units plus the cube of the units.*

Illustrative example: Find the cube root of 103,823.

Solution: First divide the number into periods of three figures each beginning at the decimal point.

$$
\begin{array}{rl|l}
 & & 103'823\,|\,47 \\
 & & 64\,000 = t^3.\ t = 40. \\
\text{Trial divisor} = 3t^2 = 3 \cdot 40^2 = & 4800 & 39\,823 \\
3tu = 3 \cdot 40 \cdot 7 = & 840 & \\
u^2 = 7^2 = & 49 & \\
\text{complete divisor} = & 5689 & \\
7 \cdot 5689 = & & 39\,823
\end{array}
$$

In order to work out the cube root of any number you should remember

1. *To divide the number into periods of three figures each from the decimal point.*

2. *To obtain the largest cube root of the cube found in the first period at the left, and write this cube root as the first figure of the result.*

3. *Take three times the square of the part of the result already found and annex two zeros as the trial divisor.*

4. *Divide this trial divisor into the remainder* (39823 above example) *and annex the integral part of this quotient to the root already obtained.*

5. *Add three times the product of the new root figure by the part of the root already found, with one zero annexed, and also the square of the new root figure, to the trial divisor to form the complete divisor.*

6. *Multiply this complete divisor by the new root figure and subtract this result from the remainder.*

7. *Continue in this manner until the cube root or the required number of decimal places are obtained.*

Exercises

Find the cube root of the following numbers:

1. 157,464.
2. 2744.
3. 262,144.
4. 132,651
5. 46,656
6. 804,357.
7. 1601.613
8. 820.025856
9. 426.957777
10. 56.623104
11. 2.5
12. .2
13. 104.

Find to three decimal places the cube root of

Find the sixth root of

14. 531441.

180. Proportion Transformations. — A proportion may be transformed in various ways.

1. If $a : b = c : d$

Then $\frac{a}{b} = \frac{c}{d}$ and $ad = bc$.

The product of the means equals the product of the extremes.

Or $\frac{ad}{ac} = \frac{bc}{ac}$. By dividing each term by ac.

Whence $d : c = b : a$

Or $b : a = d : c$.

Note that this is the same form as we would obtain by inverting the terms of the original proportion. It is called transformation by *inversion.*

If $\frac{5}{10} = \frac{6}{12}$, does $\frac{10}{5} = \frac{12}{6}$?

2. Similarly by dividing each term of $ad = bc$ by cd we may obtain

$$a : c = b : d.$$

Here the terms of the given proportion are alternated, and this is called transformation by *alternation.*

If $\frac{2}{3} = \frac{4}{6}$, does $\frac{2}{4} = \frac{3}{6}$?

3. Given $\frac{a}{b} = \frac{c}{d}$.

Then $\frac{a}{b} + 1 = \frac{c}{d} + 1$. By adding 1 to each member.

Whence $\frac{a+b}{b} = \frac{c+d}{d}$.

This is called transformation by *addition*.

If $\frac{2}{5} = \frac{8}{20}$, does $\frac{2+5}{5} = \frac{8+20}{20}$?

4. Similarly if $\frac{a}{b} = \frac{c}{d}$

And $\frac{a}{b} - 1 = \frac{c}{d} - 1$. By subtracting 1 from each member.

Whence $\frac{a-b}{b} = \frac{c-d}{d}$.

This is called transformation by *subtraction*.

If $\frac{5}{3} = \frac{10}{6}$, does $\frac{5-3}{3} = \frac{10-6}{6}$?

5. By combining the results of (3) and (4) we may obtain

$$a + b : a - b = c + d : c - d.$$

This is called transformation by *addition and subtraction*.

If $\frac{3}{4} = \frac{6}{8}$, does $\frac{3+4}{3-4} = \frac{6+8}{6-8}$?

6. In a series of equal ratios, *the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Proof :

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$

Let r = the value of each of the equal ratios.

Then $\frac{a}{b} = r, \frac{c}{d} = r, \frac{e}{f} = r.$

Or $a = br, c = dr, e = fr.$

Then $a + c + e = (b + d + f)r.$

Or $\frac{a + c + e}{b + d + f} = r = \frac{a}{b}.$

If $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, then does $\frac{1 + 2 + 3}{2 + 4 + 6} = \frac{1}{2}$?

7. Other properties of a proportion are

(*a*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a^2}{b^2} = \frac{c^2}{d^2}.$

(*b*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{\sqrt[4]{a}}{\sqrt[4]{b}} = \frac{\sqrt[4]{c}}{\sqrt[4]{d}}.$

(*c*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{ha}{hb} = \frac{kc}{kd}.$

(*d*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{ha}{bk} = \frac{hc}{kd}.$

181. Applications. — There are numerous applications of the preceding principles in algebra and geometry.

Illustrative example I:

If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{3a + 4b}{3c + 4d} = \frac{3a - 4b}{3a - 4d}.$

Proof:

1. Since $\frac{a}{b} = \frac{c}{d}$, then $\frac{3a}{4b} = \frac{3c}{4d}.$ Prin. 7 (*d*).

2. Then $\frac{3a + 4b}{3a - 4b} = \frac{3c + 4d}{3c - 4d}.$ Prin. 5.

3. Whence $\frac{3a + 4b}{3c + 4d} = \frac{3a - 4b}{3c - 4d}.$ Prin. 2.

Illustrative example II:

Transform the following proportion in order that only one term shall contain x.

$$6 : 5 = 15 - x : x$$

Solution: $\dfrac{6+5}{5} = \dfrac{15-x+x}{x}$. Prin. 3.

Illustrative example III:

Solve $\dfrac{x^2+x-3}{x-3} = \dfrac{3x^2+4x-5}{4x-5}$.

Solution: $\dfrac{x^2}{x-3} = \dfrac{3x^2}{4x-5}$. Prin. 4.

Whence $x^2[4x - 5 - 3(x-3)] = 0$.

And, $x = 0, 0, -4$.

Illustrative example IV:

If $(3a + 6b + c + 2d)(3a - 6b - c + 2d)$
$= (3a - 6b + c - 2d)(3a + 6b - c - 2d)$

Prove that a, b, c, and d are in proportion.

Solution:

$\dfrac{3a+6b+c+2d}{3a-6b+c-2d} = \dfrac{3a+6b-c-2d}{3a-6b-c+2d}$. From product of means = product of extremes

$\dfrac{2(3a+c)}{2(6b+2d)} = \dfrac{2(3a-c)}{2(6b-2d)}$. Prin. 5.

$\dfrac{3a+c}{3a-c} = \dfrac{6b+2d}{6b-2d}$. Prin. 2.

$\dfrac{6a}{2c} = \dfrac{12b}{4d}$. Prin. 5 again.

Whence $a : b = c : d$.

Exercises

1. Write by alternation

(*a*) $\frac{4}{5} = \frac{20}{25}$. (*b*) $\frac{3}{4} = \frac{a}{x}$. (*c*) $\frac{c}{d} = \frac{x}{y}$.

2. Write these same proportions by inversion.

3. Write these same proportions by addition.

4. If $\frac{a}{b} = \frac{h}{k}$, prove that $\frac{a+b}{h+k} = \frac{b}{k}$.

5. If $\frac{a}{b} = \frac{d}{e}$, prove that $\frac{a-b}{d-c} = \frac{b}{e}$.

6. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{3a-4b}{b} = \frac{3c-4d}{d}$.

Transform the following proportions in order that only one term shall contain x.

7. $b : 3 = 4 - x : x$.

8. $15 : 5 = 3 + x : x$.

Solve the equations

9. $3x - 1 : 6x - 7 = 7x - 10 : 9x + 10$.

10. $x^3 - 3x^2 + 5 : x^3 + 3x^2 - 5 = x^2 + 2 : x^2 - 2$.

11. $\frac{x^2 - 2x + 3}{2x - 3} = \frac{x^2 - 3x + 5}{3x - 5}$.

12. $x + 5 : 2x - 3 = 5x + 1 : 3x - 3$.

13. $\sqrt{x} + \sqrt{b} : \sqrt{x} - \sqrt{b} = a : b$.

14. $x^2 - 4x + 2 : x^2 - 2x - 1 = x^2 - 4x : x^2 - 2x - 2$.

If $a : b = c : d$, prove that

15. $a + 2b : b = c + 2d : d$.

16. $ma + nb : ma - nb = mc + nd : mc - nd$.

17. $ma^2 + nc^2 : mb^2 + nd^2 = a^2 : b^2$.

If $a : b = b : c$, prove that

18. $a + b : b + c = a : b$.

19. $a : c = (a + b)^2 : (b + c)^2$.

20. Find the value of x and y when

$x + y + 1 : x + y + 2 = 6 : 7$ and when

$y + 2x : y - 2x = 12x + 6y - 3 : 6y - 12x - 1$.

21. Find two numbers, such that if 7 be added to each they will be in the ratio 2 to 3; and if 2 be subtracted from each, they will be in the ratio of 1 to 3.

22. If $a : b = c : d$, prove that $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

23. Given a triangle DEF with AB drawn parallel to EF, then $AD : AE = DB : BF$.

What does this proportion become by alternation? By addition? By inversion?

24. If $AD = 14$, $AE = 7$, and $DB = 15$, find the value of BF.

25. If $DE = 20$, $AD = 12$, and $DF = 26$, find DB.

Hint: Let $DB = x$ and $BF = 26 - x$.

CHAPTER XVIII

REVIEW

182. Miscellaneous Review. — Throughout the book we have followed the plan of our First Course in Algebra, of closing many lists of exercises with several review examples. If these are used consistently, it will make an extended review at the end of the course unnecessary. Some general review is, however, often desirable, and the following exercises are offered as furnishing abundant material for such review.

Exercises

1. Find a mean proportional between

$$\frac{x^2 - x - 6}{x + 4} \text{ and } \frac{x^2 + x - 12}{x + 2}.$$

2. Solve as a quadratic $\sqrt{x + 14} - \sqrt[4]{x + 14} = 2$.

3. Two numbers are in the ratio of 7 to 9, but if 14 is added to each, they will be in the ratio of 5 to 6. Find the numbers.

4. Determine graphically the values of x and y in

$$8x - 3y = -6.$$
$$4x + 6y = 7.$$

5. Prove that in an arithmetical progression $S = \frac{n}{2}(a+l)$.

6. Solve $3\sqrt{x} - \sqrt{x + 12} = \frac{8}{\sqrt{x + 12}}$.

7. Explain the meaning of a negative integral exponent. Of a fractional exponent.

8. If 175 apples lie in a straight line at intervals of 10 feet, find the distance a boy will travel to gather up the apples, assuming that he starts from the first and makes a separate journey for each of the others.

9. Simplify each of the following: $\sqrt{-6} \times \sqrt{-80}$; $3\sqrt{-20} - 3\sqrt{-45} + \sqrt{-80}$; $\sqrt{63} \div (-\sqrt{-7})$.

10. Solve $x^3 - y^3 = 19.$
$x^2y - xy^2 = 6.$

11. Solve as a quadratic
$$x^2 + 2x + 10 - \sqrt{x^2 + 2x + 10} = 20.$$

12. A laborer received \$15 for a certain number of days' work. If he had received 25 cents less a day, it would have taken him 2 days longer to earn the same amount. How long did he work?

13. Solve $x + y + \sqrt{x + y} = 12.$
$x - y + \sqrt{x - y} = 2.$

14. Factor each of the following:
$$x^4 + 4,\ m^2 - 2mn + n^2 + 5m - 5n,$$
$$a^2 - n^2 - m^2 - 2ab + 2mn + b^2,$$
$$x^5 + 7x^4 - 49x^2 - 7x^3 + 6x + 42.$$

15. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}} - 3a^{\frac{1}{4}} + 1 - a^{-\frac{1}{4}} + a^{-\frac{1}{2}}$ by $a^{\frac{1}{4}} - 2 + a^{-\frac{1}{4}}$.

16. Simplify each of the following: $(3\sqrt[4]{\tfrac{1}{9}})^2$, $\sqrt[3]{\frac{a}{bc}}$, $\sqrt{\frac{a}{bd}}$, $\frac{3\sqrt{11}}{2\sqrt{98}}$, $7\sqrt{2\tfrac{2}{5}}$, $\sqrt{24} + \sqrt{54} - \sqrt{6}$.

17. In a geometrical progression $a = 2$, $r = \frac{1}{2}$, $l = \frac{1}{16}$. Find S and n.

18. What two numbers whose difference is d are to each other as $a : b$?

19. Reduce each of the following to its simplest form with positive exponents: $a^8 \cdot a^{-2}$, $(3\,a)^{-5} \times (3\,a)^5$, $a^{m-3} \cdot a^{3-2m}$, $\dfrac{8\,a^{-x}b^{-y^2}}{a^{3\,x}b^{5\,y^2}}$.

20. Solve $\dfrac{ax+b}{cx+d} = 1$. Determine the value of x when $a = 3, b = 2, c = 1$, and $d = 0$.

21. Simplify $\dfrac{3+2\sqrt{-1}}{3-2\sqrt{-1}}$, $(2+3\,i)(3-4\,i)$.

22. Find the value of $x^2 - 6\,x + 14$ if $x = 3 + \sqrt{-5}$.

23. Without extracting the roots determine which is the greater, $\sqrt{7}$ or $\sqrt[3]{18}$.

24. Free the following equation from radicals and find the value of x when $q = 0$; $x = -\frac{1}{2}\,q + \sqrt{-\frac{3}{4}\,q^2 + \sqrt{q^4 + r^4}}$.

25. The hypotenuse of a right triangle is 20. The sum of the other two sides is 28. Find the length of the sides.

26. In a geometric progression the sum of the first and second terms is 12 and the sum of the third and fourth terms is 108. Find the ratio, and the sum of the first 7 terms.

27. The sum of all the even integers from 2 to a certain number inclusive is 702. Find the last of these integers.

28. Plot the graph of the following system of equations,

$$x^2 + y^2 = 4.$$
$$3\,x - 2\,y = 6.$$

and from the graphs find the approximate values of x and y that satisfy both equations.

29. Find the roots of the equation $ax^2 + bx + c = 0$ and discuss their values when c and a are both positive.

30. Solve as a quadratic $\left(x - \dfrac{1}{x}\right)^2 + \dfrac{5}{6}\left(x - \dfrac{1}{x}\right) = 1$.

31. Simplify $\dfrac{\sqrt{-75}}{\sqrt{-3}}$, $\sqrt{-a^2}+\sqrt{-b^2}+\sqrt{-c^2}$, $\sqrt{-8}\cdot(-\sqrt{-18})\cdot\sqrt{-32}$, $\dfrac{-\sqrt{25}}{\sqrt{-5}}$.

32. Expand $(a^2+2b^2)^4$ by the binomial formula. Prove the expansion when $a=1$ and $b=1$.

33. Multiply $a^2b^{-2}+2+a^{-2}b^2$ by $a^2b^{-2}-2-a^{-2}b^2$.

34. Solve as a quadratic $x^4-4x^3-10x^2+28x-15=0$.

35. The product of four terms of a geometric series is 4 and the fourth term is 4. Find the series.

36. Insert four arithmetical means between a and 1 and find the sum of these four means.

37. Find x when $\sqrt{x}+\sqrt{b}:\sqrt{x}-\sqrt{b}=a:b$.

38. A launch can go 24 miles down stream and return, making the round trip in 9 hours. If the rate of the stream is 2 miles an hour, find the rate of the launch in still water.

39. Find two numbers whose sum, whose product, and the difference of whose squares are all equal.

40. Plot the graph of $x^2+3x-4=0$ and from the graph determine the roots of the equation.

41. What relation must exist between the coefficients of a quadratic equation if the sum of the reciprocals of the roots is equal to unity?

42. Solve $x^4+x^2y^2+y^4=21$.
$x^2-xy+y^2=3$.

43. Simplify $x^{\frac{3}{2}}-6x+15x^{\frac{1}{2}}-20+15x^{-\frac{1}{2}}-6x^{-1}+x^{-\frac{3}{2}}$, when $x=4$.

44. Simplify $\dfrac{3\sqrt{-5}-\sqrt{-2}}{2\sqrt{-5}-3\sqrt{-2}}$.

45. Simplify $\sqrt{50}+\sqrt[6]{9}-4\sqrt{\tfrac{1}{2}}+\sqrt[3]{24}+\sqrt[9]{27}-\sqrt[4]{64}$.

46. Using the binomial formula, write the first five terms of $(m^2 - n^2)^6$.

47. Solve and check $\sqrt{x + a} + 2\sqrt{x + 6\,a} = \dfrac{16\,a}{\sqrt{x + a}}$.

48. Solve $x^2 + y^2 + 2\,x + 2\,y = 23.$
$xy = 6.$

49. A man travels 445.5 miles. He goes 10 miles the first day and increases his speed $\frac{1}{2}$ mile each succeeding day. How many days does he travel?

50. The denominator of a certain fraction exceeds the numerator by 2. If a certain number be added to both numerator and denominator, the value of the fraction thus formed is $\frac{2}{3}$, while if this same number is subtracted from both numerator and denominator, the value of the fraction formed is $\frac{1}{2}$. Find the original fraction.

51. Without solving, determine the character of the roots of the equation $3\,x^2 + 9\,x - 12 = 0$. Give the reason for each statement concerning these roots.

52. The difference of the cubes of two numbers is 342. If the product of the numbers is multiplied by their difference the result is 42. Find the numbers.

53. The sum of three numbers in the ratio $5 : 6 : 8$ is 57. Find the numbers.

54. Factor the following: $49\,a^2 + 4 - 36\,b^2 - 28\,a$, $a^4 + 4\,b^4$, $m^8 - m^5 + 32\,m^3 - 32$, $(1 + x^3) + (1 + x)^3$, $8\,x^2 - 14\,bx + 3\,b^2$.

55. Prove that a^{-1}, a^0, and $a^{\frac{3}{4}}$, equal respectively $\dfrac{1}{a}$, 1, and $\sqrt[4]{a^3}$.

56. If $x - y$ is a mean proportional between y and $y + z - 2\,x$, prove that x is a mean proportional between y and z.

57. Determine without extracting the roots which one of the following is the greatest: $\sqrt[3]{10}$, $\sqrt{6}$, $\sqrt[4]{17}$.

58. Solve $x\sqrt{x} - 2\sqrt{x} = x$.

59. Solve $2x + \sqrt{xy} = 10$.

$$3y - 2\sqrt{xy} = -1.$$

60. Plot the graph of $x^2 - 4x + 3 = y$.

61. In two years the population of a city increased from 64,000 to 81,000. The rate per cent of increase during the first year was equal to the rate per cent of increase during the second year. What was this rate?

62. Separate the number n into 3 parts such that the first is to the second as a is to b and the second is to the third as c is to d.

63. Find the square root of

$$5x^2 - 23x^4 + 12x + 8x^5 - 22x^3 + 16x^6 + 4.$$

64. Show that the fraction $\dfrac{x^n - \sqrt{x^{2n} - 4}}{2}$ is the reciprocal of the expression $\left(\dfrac{\sqrt{x^n + 2} + \sqrt{x^n - 2}}{2}\right)^2$.

65. Simplify:

$$\frac{\dfrac{a^2 + b^2}{b} - a}{\dfrac{1}{b} - \dfrac{1}{a}} \times \frac{a^2 - b^2}{a^3 + b^3} \times \left(\frac{a + b}{a - b} + \frac{a - b}{a + b}\right)\left(\frac{a}{a + b} + \frac{b}{a - b}\right).$$

66. Write in the simplest form the quadratic equation whose roots are $a + b \pm (a - b)\sqrt{-1}$.

67. Simplify:

$$\frac{m^2 + 2mn + n^2}{2m - 3n} \times \frac{8m^2 - 18n^2}{3m^3 - 24n^3} \div \frac{2m + 3n}{m^2 - 4n^2}.$$

68. Derive the formula for the sum of n terms of an arithmetical progression whose first term is a and whose common difference is d.

69. Two numbers are to each other as 3 to 2. If 6 is added to the greater and subtracted from the less, the sum will then be to the difference as 3 is to 1. Find numbers.

70. Solve $2\sqrt{5x} - \sqrt{2x-1} = \dfrac{4x+1}{\sqrt{2x-1}}$.

71. Compare the magnitudes of $\sqrt[3]{\frac{3}{2}}$ and $\sqrt{\frac{4}{3}}$.

72. Determine under what conditions the roots of the equation $ax^2 + bx + c = 0$ will be (a) real and equal, (b) rational and unequal, (c) imaginary and unequal.

73. Find four numbers in geometric progression such that the sum of the first and fourth is 56 and the sum of the second and third 24.

74. Simplify each of the following:

$$-\sqrt{18} \div \sqrt{-6}, \quad (\sqrt{-4} + \sqrt{2})(2\sqrt{-4} + 3\sqrt{2}),$$

$$2\sqrt{-16} - 5\sqrt{-49} + 3\sqrt{-81}.$$

75. Determine graphically the roots of $x + y = 5$.
$x + 2y = 4$.

76. Factor each of the following:

$a^2 + b^2 - c^2 - 9 - 2ab + 6c$. $a^4 - 3a^2 + 9$.

$2y + x^2 - 4y^2 - x$. $x^6 + y^6$.

77. Find to three terms the square root of $x^2 - 6$.

78. Separate 12 into two parts such that their product shall be to the sum of their squares as 2 is to 5.

79. From $\dfrac{3}{5 - 2\sqrt{3}}$ subtract $\dfrac{1}{2 + \sqrt{3}}$ and express the result as a fraction having a rational denominator.

80. Solve $2x^{\frac{1}{3}} = x^{\frac{1}{6}} + 1$.

81. Write with integral coefficients the quadratic equation whose roots are $\frac{2+\sqrt{-11}}{3}$ and $\frac{2-\sqrt{-11}}{3}$.

82. Prove that in a geometrical progression $S = \frac{a(1-r^n)}{1-r}$.

83. Divide:

$9\,a^{\frac{1}{2}} - 3\,a^{\frac{1}{4}} + 1 + 7\,a^{-\frac{1}{4}} - 6\,a^{-\frac{1}{2}}$ by $3 + a^{-\frac{1}{4}} - 2\,a^{-\frac{1}{2}}$.

84. Write the rth term of $(a-b)^n$. Apply this formula to finding the sixth term of $(2\,m^2 - x^{\frac{1}{2}})^8$.

85. Solve $\frac{1}{cx} + \frac{1}{dy} = m.$

$$\frac{1}{dx} + \frac{1}{cy} = n.$$

86. Plot the graph of $x^2 - 2\,x - 2 = y$ between the values $x = -3$ and $x = +4$ and estimate from the graph the roots of $x^2 - 2\,x - 2 = 0$.

87. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$ when the sum of the means is $2\frac{1}{2}$?

88. Solve $x^{-4} - 4\,x^{-2} - 21 = 0$.

89. Find the sum of the series $4, -\frac{8}{3}, \frac{16}{9}, \cdots$ to infinity.

90. Simplify:

$$\left(x^2 - y^2 + \frac{4\,xy(y+x)}{x-y}\right) \div \left(\frac{x^2 + y(y+2\,x)}{4\,x^2 - 6\,xy + 2\,y^2}\right).$$

91. Find two numbers in the ratio of 3 to 2 such that their sum shall be to the sum of their squares as 1 is to 5.

92. Write with integral coefficients the quadratic equation whose roots are $\frac{1}{2}(3+\sqrt{-7})$ and $\frac{1}{2}(3-\sqrt{-7})$.

93. Solve $x^2 + px + q = 0$, and state the relation between the roots and the coefficients.

94. Solve $3\sqrt{x+8} - \sqrt{x-8} = 2\sqrt{2\,x+2}$.

95. The sum of the cubes of two numbers is 756. The sum of the squares of these numbers diminished by their product is equal to 63. Find the numbers.

96. A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money, he would have paid \$15 less for each piece. Find the number of pieces purchased.

97. Solve graphically the following, estimating the roots to the nearest tenth.

$$y = 8 - x.$$
$$2x - 6y = -9.$$

98. Divide $\dfrac{1}{x^{-\frac{3}{5}}} - \dfrac{1}{y^{-\frac{3}{5}}}$ by $\sqrt[5]{x} - \sqrt[5]{y}$.

99. Find the prime factors of

$(x^2 + x - 2)^2 - (x^2 - x + 3)^2$, $12xy + 25 - 4x^2 - 9y^2$, $12a^2 - 4ab - 3ax^2 + bx^2$, $64 - n^6$.

100. (a) Simplify $6 \div \sqrt{3} + \sqrt{2}$.

(b) Multiply $2\sqrt{a} + 5\sqrt{a-b}$ by $\sqrt{a} - \sqrt{a-b}$.

101. Perform the indicated operations and express the results in their simplest forms: $\sqrt{-25x^2} - \sqrt{-4x^2} - \sqrt{-9x^2} - \sqrt{-1}$, $(\sqrt{-3} + \sqrt{-5})^2$, $\dfrac{2 + 3\sqrt{-1}}{2 + \sqrt{-1}}$.

102. Solve $x^4 + x^2y^2 + y^4 = 91$.
$$x^2 - xy + y^2 = 7.$$

103. In an arithmetical progression the first term is $\frac{1}{2}$, the last term is $-\frac{5}{11}$, and the number of terms is 22. Find the difference between any two successive terms.

104. Solve $x^{-\frac{4}{3}} - 5x^{-\frac{2}{3}} + 4 = 0$.

105. Solve graphically $x^2 - 4y = 18$.
$$3x = 8 + 4y.$$

106. A number of men subscribed a certain amount to make up a deficit of \$100, but 5 men failed to pay and thus increased the share of the others by \$1 each. Find the amount each paid.

107. Factor, find the highest common factor and the lowest common multiple of $a^4 + a^2b^2 + b^4$, $4a^3 + 4b^3$, $2a^2c - 2abc + 2b^2c$.

108. Simplify each of the following expressions:

$$\sqrt{3} \times \sqrt[6]{2} \times \sqrt[3]{5}, \quad \frac{\sqrt{-6}}{-3\sqrt{-2}}, \quad \frac{3+\sqrt{-2}}{2-\sqrt{-2}}.$$

109. Reduce to lowest terms $\dfrac{2x^3 + x^2 - 25x + 12}{3x^3 + 5x^2 - 34x - 24}$.

110. (a) Solve by the graphic method the equation

$$x^2 = 2x + 3.$$

(b) Solve by the formula the equation $x^2 + x + 1 = 0$.

111. If the series $\frac{2}{5}, \frac{9}{10}, \cdots$ be arithmetical, find the sum of the first 15 terms; if geometrical, find the 5th term.

112. Determine, without solving, the nature of the roots of the equation $3x^2 - 5x + 3 = 0$.

113. (a) If $x^{-\frac{2}{3}} : 2 = 1 : x^{\frac{1}{6}}$, what is the value of x?

(b) Simplify and express with positive exponents

$$\frac{a^{-1}b\sqrt{c}}{a^{\frac{2}{3}}} \div \frac{\sqrt{a^2b^{-1}}}{c^{-3}}$$

114. (a) Form the quadratic equation whose roots are 5 and $\frac{2}{3}$.

(b) State the relation between the roots and the coefficients in a quadratic equation.

115. A merchant bought a number of barrels of apples for \$120. He kept two barrels and sold the remainder at an advance of \$2 per barrel, thereby gaining \$34. How many barrels did he buy?

116. A train traveled 273 miles at a uniform rate. If the rate had been 3 miles an hour less, the journey would have taken $\frac{1}{2}$ hour longer. Find the rate of the train.

117. Find the square root of $\frac{a^2}{x^2} + \frac{6\,a}{x} + 11 + \frac{6\,x}{a} + \frac{x^2}{a^2}$.

118. Solve and check $2\sqrt{x-a} + 3\sqrt{2\,x} = \frac{7\,a + 5\,x}{\sqrt{x-a}}$.

119. Solve and check $4\,x^2 + 5\,x + 2 = 0$.

120. Solve $x^2 + y - 8 = 0$.
$x^2 + 15\,y - 16 = 0$.

121. Factor $x(x+1)(x-2) - 3\,x - 3$, $a^4 + 4$.

122. On a certain day the temperature rose $\frac{1}{2}$ degree hourly from 5 A.M. to 11 A.M., and the average temperature for that time was 8°. What was the temperature at 9 A.M.?

123. Find the common difference of the arithmetical progression whose 1st term is 3 and whose 2d, 4th, and 8th terms are in geometrical progression.

124. Without solving, show whether the roots of the following quadratics are (1) real and equal, (2) real and unequal, or (3) imaginary.

(a) $2\,x^2 - 4\,x - 7 = 0$.
(b) $3\,x^2 - 4\,x + 5 = 0$.
(c) $-x^2 + 9\,x - 1 = 0$.
(d) $12\,x^2 - 84\,x + 147 = 0$.

125. Expand and simplify, expressing the result with positive exponents: $(2\,a^{-\frac{2}{3}} - a^{\frac{1}{3}}b^2)^4$.

126. A rectangular field is 119 yards long and 19 yards wide. How many yards must be added to its width and how many taken from its length, in order that its area may remain the same while its perimeter is increased by 24 yards?

127. Plot on the same sheet and with the same axes

(a) $y = 3\,x - 2$.
(b) $y = -2\,x + 4$.
(c) $y = -x + 5$.

Find from the graph the solution of $y = -2x + 4$.

$y = 3x - 2$.

128. Express in simplest form with positive exponents the first four terms of the following: $(a^{\frac{2}{3}}b^{-\frac{1}{2}} - 2a^{-\frac{1}{3}}b^{\frac{3}{2}})^6$.

129. Solve the following equation, finding all four roots:

$$(x^2 - 6x)^2 - 2(x^2 - 6x) = 35.$$

130. (a) Find the square of $-\frac{5}{2} - \frac{5}{2}\sqrt{-3}$.

(b) Simplify $\dfrac{2 - 3\sqrt{-1}}{3 - \sqrt{-4}} + \dfrac{2 + \sqrt{-9}}{3 + 2\sqrt{-1}}$.

131. Solve $x^2 - y^2 = 21$.

$2x^2 - xy - y^2 = 36$.

132. Find the prime factors and indicate the H. C. F. and L. C. M. of $n^3 - 5n^2 + 2n + 8$ and $2n^3 - 3n^2 - 3n + 2$.

133. A quadratic equation has its absolute (or constant) term printed by mistake as 32 instead of 27, and the roots are found to be -8 and -4. What are the roots of the equation when it is correctly printed?

134. How much does a teacher earn in 25 years if she receives a salary of \$720 the first year and an increase of \$80 each year for 14 years?

135. By drawing the graphs, estimate as nearly as you can the solutions of the pair of equations

$$3x + 2y - 6 = 0.$$
$$y^2 - 2x = 9.$$

136. (a) Given the equation $x^2 - 3x - 28 = 0$. Write the equation the sum of whose roots is the same as that of the given equation, and the product of whose roots is two greater than that of the given equation.

(b) Find the sum and the product of the roots of the following: $2x^2 + 5ax - 6a^2 = 0$.

137. Find two numbers whose difference is 32 and whose arithmetic mean exceeds their geometric mean by 4.

138. A boy was sent with \$2.10 to buy oranges. He found the price 3¢ per dozen more than he expected, so he bought $\frac{1}{2}$ dozen fewer than he intended to buy and received 1¢ in change. What price did he pay for the oranges?

139. Solve $(x-3)^{-\frac{2}{3}} + 3(x-3)^{-\frac{1}{3}} = 4$.

140. Factor $27^{-1}x^3 - 8$.

Find the value of $\dfrac{1}{8^{-\frac{2}{3}}} - 3 \times 2^0 + 27^{-\frac{1}{3}}$.

141. Solve the following:

$$\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} = \frac{2\sqrt{x}}{\sqrt{a}+\sqrt{x}} - \frac{(x+a)^2}{a(x-a)}.$$

142. Form the quadratic equation the product of whose roots shall be twice the sum of the roots of $x^2 - 3x + 8 = 0$, and the sum of whose roots shall be twice the product of the roots of $2x^2 + 4x - 7 = 0$.

143. Plot the graph of $x^2 + y^2 = 28$ and of $5x + 2y = 10$ and from these graphs estimate the solutions of these equations taken as a system.

144. When 3, − 6, etc., is an arithmetic progression, find the sum of the first 50 terms. When it is a geometric progression, find the 12th term.

145. (a) In the quadratic equation $8x^2 + px + 9 = 0$, one root is double the other. Find p.

(b) Form the equation whose roots are $2 - \sqrt{-3}$ and $2 + \sqrt{-3}$.

146. Solve $x^2 + x + 2 = 7\sqrt{x^2 + x + 2} - 10$.

147. Write in simplest form with positive exponents the 4th and 5th terms of $(2a^{-\frac{1}{2}}b - a^{\frac{1}{2}}b^3)^8$.

148. Solve the equation $ax^2 + bx + c = 0$. State the condition under which the roots of the above equation are (1) rational, (2) unequal and containing surds, (3) complex, (4) equal.

149. The length of a sheet of paper is 3 inches greater than its width. From each corner a piece one inch square is cut out. The paper is then folded to form a box with open top. If the box contains 54 cubic inches, what were the dimensions of the sheet of paper?

150. Perform the indicated operations:

$$\left(2a - \frac{a^2 - b^2}{a}\right)\left(3b + \frac{a^2 + b^2}{b}\right) \div \left(\frac{a^2}{b^2} + 5 + \frac{4b^2}{a^2}\right).$$

151. Find the square root of the following:

$$(x^2 + 3x + 2)(x^2 - 1)(x^2 + x - 2).$$

152. Find the value of $\dfrac{1}{8^{-\frac{2}{3}}} - 3a^0 + 27^{-\frac{1}{3}} - 1^{\frac{3}{2}}$.

153. Simplify $\dfrac{a - a^{-1}}{a^{\frac{1}{3}} - a^{-\frac{1}{3}}} - \dfrac{a + a^{-1}}{a^{\frac{1}{3}} + a^{-\frac{1}{3}}}$.

154. Find the 5th term of $\left(\dfrac{a^2}{2} - a^{-1}b^{-1}\right)^8$.

155. Find the value of $\dfrac{8 - \sqrt{3}}{2 - \sqrt{3}}$ correct to three decimal places.

156. Solve $(x^2 + x - 2)^2 + (x^2 + x - 2) - 20 = 0$.

157. (a) Form the equation whose roots are $3 - \sqrt{5}$ and $3 + \sqrt{5}$.

(b) Form the equation the sum of whose roots is one greater and the product of whose roots is two less than the sum and product respectively of the roots of

$$x^2 + 5x - 3 = 0.$$

158. If the first two terms of an arithmetic progression are 5 and 3, write the next two terms. Find by formula the 8th term. Find the sum of 15 terms.

159. If the first two terms of a geometric progression are 5 and 3, write the next two terms. Find by formula the 8th term. Find the sum to infinity.

160. Draw the graph of $x^2 - 5x + 3 = y$, and thus determine approximately the roots of $x^2 - 5x + 3 = 0$.

161. Find the arithmetic progression whose 1st term is 3 and whose 1st, 3d, and 6th terms form a geometric progression.

162. An open box is made from a square piece of tin by cutting out a 6-inch square from each corner and turning up the sides. The box contains 150 cubic inches. Find the area of the original square.

163. (a) What must be the value of k in order that the roots of $2x^2 - kx - 5 = 0$ shall be $\frac{1}{2}$ and -5?

(b) Find the values of m for which the roots of $x^2 - mx + m + 3 = 0$ are equal.

164. Perform the indicated operations and simplify the result:

$$\left(2 + \frac{m}{m-3}\right)\left(\frac{9-m^2}{4-m^2}\right)\left(\frac{m+2}{m^2+m-6}\right) - \left(\frac{2}{m+2}\right).$$

165. Solve and check:

$$m^2 - 8m + 40 - 2\sqrt{m^2 - 8m + 40} = 35.$$

166. (a) Solve the formula for the value of L:

$$C = \frac{E}{\sqrt{R^2 + (2\pi nL)^2}}.$$

(b) Determine which of the following surds is the greater, $\sqrt[3]{3}$ or $\sqrt[5]{6}$.

167. Find the 11th term and the sum of the first 11 terms of the progression $1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$.

168. Factor the following, $12\,a^2 - 5\,ab - 2\,b^2$, $x^3 - 6\,x^2 + 11\,x - 6$, $16\,a^2b^2 - (a - b)^4$, $16\,m^4 - 9\,m^2n^2 + 25\,n^4$.

169. Simplify the following and express the result with positive exponents:

$$\sqrt{(a^{-1}c^{-1}p^2qr^{-3})(a^{-1}b^3p^{-3}q^2r)(bc^{-3}p^{-1}q^3r^{-2})}.$$

170. Solve and check $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 61.$

$$\frac{1}{x} - \frac{1}{y} = 1.$$

171. The sum of the areas of two square pieces of tin is 25 square feet, and a side of the larger is 1 foot longer than a side of the smaller. State the equations in two unknown quantities and find graphically the length of a side of each square.

172. An automobile is driven at the rate of 20 miles an hour for 4 hours. After a stop of 2 hours the run is resumed for 3 hours at the same rate, when an hour's stop is made. After another run of an hour and a half at the same rate the automobile reaches its destination. Construct a graph to represent the journey and from it determine how far the automobile is from its starting point at the end of the eighth hour.

173. A stone dropped from a floating balloon takes 10 seconds to reach the surface of a lake below. If the numbers in the series 16, 48, 80 . . . represent the number of feet the stone falls in successive seconds, how high is the balloon above the lake?

174. There are two alloys of silver and copper, one of which contains twice as much copper as silver, the other three times as much silver as copper. How much of each alloy must be taken to make an alloy of equal parts of silver and copper which shall weigh $2\frac{1}{2}$ pounds avoirdupois?

175. (a) Solve $2x^2 + 4 = 5x$ and check by obtaining the coefficient of x and the constant term from the roots obtained.

(b) By using the discriminant $b^2 - 4ac$, determine which of the following have rational factors:

$$20x^2 - 9x - 20; \quad 13x^2 + x - 12.$$

176. Find the prime factors of the following:

$18a^2 + 27a - 5.$ $\qquad x^2 - 6x + 9 - y^2.$

$x^3 - 7x - 6.$ $\qquad a^4 + 4a^2 + 16.$

177. Extract the square root of

$$4x^{2m} + 4x^m - 3 - 2x^{-m} + x^{-2m}.$$

178. Solve for x and check: $\sqrt{7x + 1} = 1 + \sqrt{3x + 10}$.

179. (a) Solve for d in terms of l, H, and V:

$$V = \sqrt{\frac{H \cdot 2500}{l \cdot \frac{13.9}{d}}}.$$

(b) What is the total surface of a rectangular block if its length is l'', its width is w'', and its height is h''?

180. Solve $\dfrac{x-5}{x+3} - \dfrac{x-8}{3-x} = \dfrac{80}{x^2-9} + \dfrac{1}{2}$.

181. Solve for x and y either graphically or algebraically,

$$x^2 + y^2 = 25.$$
$$xy = 12.$$

182. Graph the equation $v = 32t$, letting the abscissa distances represent values of t and the ordinate distances represent the values of v. If t represents time in seconds and v velocity in feet, read from the graph the velocity of a falling body at the end of 4 seconds; at the end of 6 seconds. Read from the graph the time required to give the body a velocity of 16 feet.

183. (a) A toboggan slide of uniform grade is to be built on a perfectly level surface and is to have 10 supports, the distances between them being equal. The length of the longest support is to be 42 feet 6 inches, and the length of the shortest is to be 2 feet. How long must each of the supports be?

(b) Derive the formula used in the solution of a.

184. (a) Change the repeating decimal .7272 . . . to a common fraction.

(b) Derive the formula used in the solution of a.

185. A rectangular wheat field is 80 rods long by 60 rods wide. A strip of uniform width is cut around the field, so that half the grain is left standing in the form of a rectangular plot. How wide is the strip that is cut?

186. (a) Solve $3x^2 - 5x + 4 = 0$ and check by deriving the entire equation from the roots obtained.

(b) Determine the nature of the roots of the following equations, $4x = 9 - x^2$, $4x = 12 - x^2$, $-4x = 12 + x^2$, $-4x = 1 + 4x^2$.

187. Solve $\dfrac{a}{x} - \dfrac{b}{y} + 1 = 0.$

$$\frac{b}{x} - \frac{a}{y} + 1 = 0.$$

188. Find the dimensions of a rectangle such that its perimeter is 28 inches and its diagonal is 10 inches.

189. Simplify (a) $25^\circ + 0.25^{\frac{1}{2}} - 8^{\frac{1}{3}} \cdot 4^{-\frac{1}{2}} + 0.027^{\frac{1}{3}}$.

(b) $\sqrt[n]{\dfrac{32}{2^{5+n}}}$.

190. Plot the graph of 2^x from $x = -4$ to $x = +4$.

191. Find the value of $x^3 - 3x^2 + x + 1$ if $x = 1 - \sqrt{2}$.

192. A freight boat leaves New York, sailing for a European port at the rate of 15 knots an hour. Five hours later a transport leaves the same harbor, sailing over the same course at the rate of 20 knots an hour. In how many hours will the transport overtake the freight boat?

193. The edge of a cube is e, its volume is v, and its total surface is s. Express v in terms of e, s in terms of e, diagonal of cube in terms of e, e in terms of s, v in terms of s.

194. Three old machines completed a job in 80 minutes. After two new machines had been installed, one of which worked three times as fast as all the old machines together, the job was completed by the five machines in 15 minutes. In what time would the second new machine alone have been able to do the work?

195. Simplify $\dfrac{\dfrac{1}{a+x}+\dfrac{1}{a-x}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}}$.

196. Solve $a + x = \dfrac{1}{a} + \dfrac{1}{x}$.

197. Solve by formula and by factoring $2x^2 - 5x = 3$.

198. Solve for f the formula $\dfrac{1}{f} = \dfrac{1}{r} + \dfrac{1}{rt}$.

199. Simplify:

$$\left(\frac{3a^4 - 75a^2}{3a - 7}\right)\left(6a - \frac{7}{a} - 11\right) \div -\left(2 + \frac{5}{a^2} + \frac{11}{a}\right).$$

200. A 5% solution of salt, weighing 160 pounds, is to be reduced to a 4% solution. How many pounds of water must be added? How many gallons of water must be added, if one gallon weighs 8.35 pounds?

201. The two sides of a right triangle are 3 and 4 inches long respectively. How much must the longer leg be in-

creased in order that the hypotenuse may become 2 inches longer?

202. The cost of a certain article is 4¢ a pound. If p is the price paid for n pounds, what is the equation connecting n and p? Draw the graph of that equation.

203. Solve $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{3}$.

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{9}.$$

204. A projectile weighing w pounds, whose diameter is d inches, strikes a wrought iron plate when moving at the rate of v feet per second. The depth of penetration in inches, p, is given by the formula $p = \dfrac{v}{608.3}\sqrt{\dfrac{w}{d}} - 0.14\,d$. Find p when $d = 12.5$, $w = 1250$, and $v = 2016$.

205. An automobile, after traveling at the rate of 30 miles an hour for 3 hours, has an accident which delays it $\frac{1}{2}$ hour. It then continues its journey at the rate of 20 miles an hour. At what rate must another car, which started from the same place two hours later, travel in order to overtake the first car in 5 hours?

206. Simplify (a) $\sqrt[2n]{\dfrac{64^n}{4^{2n}}}$, (b) $\dfrac{\sqrt[3]{a^2}}{\sqrt{a^3}}$, (c) $\sqrt[3]{\dfrac{9\,v^2}{16\,\pi^2}}$.

207. Solve the equation $x^2 + 2x + k = 0$ and state for what values of k the roots are real, for what values they are imaginary, and for what value they are equal.

208. Multiply both numerator and denominator of the fraction $\dfrac{2 + \sqrt{3} + \sqrt{5}}{2 + \sqrt{3} - \sqrt{5}}$ by $2 + \sqrt{3} + \sqrt{5}$. Then rationalize the denominator of the resulting fraction.

209. Find to two decimal places the value of $\sqrt{3 \cdot 6\,\pi V^3}$ when $V = 6.281$.

210. Since 9 is a factor of 711, it is suggested that 9 may be a root of the equation $37\,x^2 = 254\,x + 711$. Without solving or substituting, prove that 9 is a root and then find the other root.

211. Determine by finding their sum and their product, whether $-\frac{1}{2} + \sqrt{2}$ and $-\frac{1}{2} - \sqrt{2}$ are the roots of the equation $x^2 + x = -1$. Give the reason for your answer.

(b) Form the equation whose roots are $-2\frac{3}{4}$ and $\frac{5}{8}$.

(c) State the sum and the product of the roots of the equation $x = 8 - x^2$.

212. Solve the following simultaneous equations:

$$3\,x^2 - 4\,y = 59.$$
$$2\,x^2 + 3\,y^2 = 98.$$

213. If a stone is thrown vertically upward with a velocity of 110 feet per second, its height in feet t seconds later is $110\,t - 16\,t^2$. Find when, if at all, the stone will be 150 feet above the ground, and when, if at all, it will be 200 feet above the ground.

214. (a) Find the hundredth term of the progression $1, 7, 13 \cdot\cdot\cdot$.

(b) Find the sum of the first 15 terms of the progression $2, 4, 8, \cdot\cdot\cdot$.

215. By the aid of graphs find one solution of the following simultaneous equations:

$$x^2 + y = 7.$$
$$x^2 + y^2 = 11.$$

216. Find the prime factors of each of the following:

$16\,y^4 - .0081$, $\quad (2\,x - 3)^2 + 3(2\,x - 3) - 10$,

$4\,a^2 - 16\,b^2 + 4\,a + 1$, $\quad x^3 + x^2 - 14\,x - 24$.

217. Writing $(1.06)^6$ as $(1 + .06)^6$, expand by the binomial formula and carry the result far enough to obtain the numerical value to two decimal places.

218. (a) Multiply, using logarithms, 27.3×0.96.

(b) Solve for n, $4 = (1.04)^n$.

219. Solve the following set of equations, correctly group your answers, and check:

$$x^2 + y^2 = 20.$$
$$x + 2y = 5.$$

220. Two trains run at uniform rates over the same 120 miles of rail. One of them travels 5 miles an hour faster than the other and takes 20 minutes less time to run the distance. Find the rate of the faster train.

221. (a) Unite into a single term

$$6\sqrt{2\tfrac{2}{3}\,a^3} - 2(24\,ab^2)^{\frac{1}{2}} + a\sqrt{54\,a}.$$

(b) Evaluate $(0.125)^{-\frac{2}{3}} + \dfrac{3}{2 + 2^{-1}}$.

(c) Evaluate 0.6×32^0, 0.8×4^{-2}, $12 \times 9^{-\frac{1}{2}}$.

222. In each of the following perform the indicated operations and write the result in its simplest form.

(a) $\dfrac{c^2 - 22}{c^2 - 2c - 8} + \dfrac{c - 5}{4 - c}$.

(b) $\dfrac{x^2 - 4x + 4}{10x - 21} \div \dfrac{x^2 + x - 6}{10x^2 + 9x - 63}$.

223. A crew can row 8 miles down a stream and back in 3 hours and 40 minutes. If the rate of the stream is $2\frac{1}{2}$ miles an hour, find the rate of the crew in still water.

224. A corner building lot, rectangular in shape, contained 9600 square feet. After a sidewalk 6 feet wide had been built on one side and the front, the area of the lot was reduced to 8316 square feet. Find the original length and width.

225. (a) Prove $\log a^p = p \log a$.

(b) By the use of logarithms find the value of

$$\frac{(.846)^2 \times \sqrt[3]{18.7}}{3.42}.$$

226. In the formula $C = \dfrac{SE}{SB + R}$ (a) Solve for S in terms of the other letters.

(b) Find the value of C to the nearest hundredth if $S = 4$, $E = 1.07$, $B = 2.4$, $R = 27$.

227. Write in simplest form

$$6\sqrt{4\tfrac{1}{6}} - \sqrt{96} - \sqrt[3]{\tfrac{2}{9}} - \sqrt{2\tfrac{7}{8}} + \tfrac{1}{3}\sqrt[3]{-48}.$$

228. Draw the graph of the equation $x^2 - 4x + 2 = y$ from $x = 1$ to $x = +5$. From the graph estimate to the nearest tenth the roots of the equation $x^2 - 4x + 2 = 0$.

229. If the cost of goods is C, the cost of doing business $b\%$ of the selling price and the profit $p\%$ of the selling price, find a formula for the selling price S.

230. Find by the use of a formula the number of consecutive integers beginning with 10 that must be taken for the sum to be 2370.

231. Find to the nearest hundredth the roots of $x^2 - 51 = 4x$.

232. A man agrees to work for a builder on the condition that for each day he works he is to receive \$6 and for each day he is idle he is to forfeit \$2. At the end of 60 days he received \$240. How many days was he idle?

233. By use of logarithms find the value of

$$\frac{.03472 \times \sqrt[3]{47.2}}{2.08}.$$

234. Find the prime factors of each of the following: $x^6 - 64$, $10a - 7a^2 - 6a^3$, $3x^{3a} + 3x^{2a} - 18x^a$, $x^3 - 8x^2 + 17x - 10$, $4(a - b)^2 - 12(a - b)c + 9c^2$.

235. (a) Rationalize the denominator in $\dfrac{\sqrt[3]{20} - \sqrt[3]{18}}{\sqrt[3]{12}}$.

(b) Find the value of $\dfrac{3^0 x + 4x^{-1}}{x^{-\frac{2}{3}}}$ when $x = 8$.

236. A labor report states that in a certain factory 1200 men and women are employed. The average daily wage is \$3.40 for a man and \$1.80 for a woman. If the labor costs \$3376 per day, how many men and how many women are employed?

237. A traveler having 18 miles to go, calculates that his usual rate would make him one half hour late for an appointment. He finds that in order to arrive on time he must travel at a rate one half mile per hour faster. What is his usual rate?

238. (a) If $10^{1.6021} = 40$, what logarithm is indicated? Why?

(b) If $\log 69 = 1.8388$, what is the value of x in $10^x = 69$? Why?

239. A certain month contained five Fridays. The sum of the numbers representing the corresponding dates was 85. On what date did the first Friday fall and on what day did the last of the month occur?

240. In each of the following equations fill in the parentheses and explain your method:

$x^2 - 4x + (\quad) = 0$, if one root is 3.

$6x^2 + (\quad)x = 12$, if one root is $\frac{4}{3}$.

$x^2 + 7 = (\quad)x$, if one root is $3 - \sqrt{2}$.

241. Solve for x and check, $2\sqrt{x} - \sqrt{4x - 11} = 1$.

242. Write as a simple fraction in its lowest terms and check, letting $a = 2$:

$$\frac{\dfrac{a-5}{2} - 7 + \dfrac{24}{a}}{\dfrac{9 - 3a}{a}}.$$

243. The formula for amount (A) when money is invested at simple interest is $A = P(1 + rt)$ where P is the principal,

r the rate of interest expressed fractionally, and t the time in years. Make a graph of the formula where $P = 200$ dollars and $r = 6\%$. From this graph estimate the number of years in which the principal will double itself.

244. Solve for x and check:

$$3\sqrt{x} - \sqrt{9x - 32} = \frac{8}{\sqrt{9x - 32}}.$$

245. In the formula $\frac{a}{g} = \frac{h - L}{h + L}$, solve for h.

In the above formula find the value of h to the nearest tenth when $a = 5.28$, $g = 16.08$, $L = 4.70$.

246. Deduce the fundamental formula for the sum of a geometric progression in terms of the first term, the ratio, and the number of terms.

247. Solve for x and y and correctly group your answers:

$$x^3 - y^3 = 26.$$
$$x^2 + xy + y^2 = 13.$$

248. Goods cost a merchant \$72. At what price should he mark them so that he may sell them at a discount of 10% from this marked price and still make a profit of 20% of the sale price?

249. Find to the nearest hundredth the side of a square whose area is doubled when its sides are increased by 2.

250. Represent graphically the equation $x^2 - 2x = y$. From this graph determine (a) the roots of $x^2 - 2x = 0$, (b) the roots of $x^2 - 2x = 6$ to the nearest tenth.

251. It cost \$74.25 to have a well dug. If the cost of digging was \$3 for the first yard, \$3.75 for the second yard, \$4.50 for the third yard, etc., how deep was the well?

252. Reduce to a simple fraction in its lowest terms and check:

$$\left(\frac{x+2}{x-2} + \frac{x-2}{x+2}\right) \div \left(\frac{x+2}{x-2} - \frac{x-2}{x+2}\right).$$

253. (a) Solve the equation $ax^2 + bx + c = 0$ by the method of completing the square.

(b) Express in simplest form the sum of the roots and the product of the roots in terms of a, b, and c.

(c) Form the equation whose roots are -7 and 3.

254. Find the roots of the equation $0.3\,x^2 = 4\,x + 1.01$ to the nearest tenth.

255. The sum of the volumes of two cubes is 43 cubic feet. Find the length of an edge of each cube if the sum of these two lengths is 4 feet.

256. Solve and check one root:

$$\sqrt{2\,x + 3} - \sqrt{x + 1} = \sqrt{5\,x - 14}.$$

257. A dealer bought a number of sheep for \$440. After 5 had died he sold the rest at a profit of \$2 each, thereby making \$60 on the whole transaction. How many did he buy?

258. Represent graphically each equation in the following set, and from the graph read the solutions common to the two equations:

$$x^2 + y^2 = 100.$$
$$x - 2\,y = -10.$$

259. Divide $6\,x^{-3} + 10\,x^{-2} - 4\,x^{-1} - 10 - 2\,x$ by $x^{-2} - 1$, and express the result with positive exponents in descending powers of x.

260. Solve for x to the nearest hundredth,

$$x^2 - 2.2\,x - 1.16 = 0.$$

261. Solve the following and correctly group your answers:

$$x^2 + y^2 = 10.$$
$$x^2 - xy + y^2 = 7.$$

262. Solve the following and check your answers.

$$\sqrt{2\,x + 2} + \sqrt{2\,x - 1} - \sqrt{8\,x + 1} = 0.$$

263. By the use of logarithms find the value of

$$\frac{1.97^6 \times \sqrt[5]{31}}{.8642}.$$

264. (a) Without solving the equation determine the nature of the roots of $6x^2 + 7x - 6 = 0$.

(b) Determine without solving or substituting whether 4 and -3 are roots of the equation

$$x^2 - x + 10 = 0.$$

265. In January a boy joins a Christmas club and plans to have \$25 by the end of the 50 weekly payments. He wishes his weekly payments to decrease by the same amount and his last payment to be 1 cent. What is his first payment, and by what amount is each payment decreased?

266. A can do a piece of work in $4\frac{1}{2}$ days. After he has worked 3 days B joins him and they finish the work in $\frac{7}{8}$ of a day. In what time can B do the work alone?

267. Represent graphically the equation $x^2 - 6x + 6 = y$ from $x = -1$ to $x = 6$ and from the graph determine to the nearest tenth the roots of the equation $x^2 - 6x + 6 = 0$.

268. Simplify $\left[\frac{\sqrt{72\,y^{2n}}}{3} \times 9^0\right](2\,y^{n+2})^{-1}$.

269. A purse contained \$6.55 in quarters and dimes. After 6 quarters and 8 dimes had been taken out, the number of quarters equaled 3 times the number of dimes. How many coins of each kind were there? (Cornell.)

270. A boat can travel 8 miles an hour in still water. If it can travel 20 miles downstream in the same time that it can travel 12 miles upstream, what is the rate of the stream? (N. Y. State Regents' Examination.)

271. Solve for t and check: $\sqrt{t+4} + \sqrt{t} = 3$. (Cornell.)

272. Find the roots of the following equation to the nearest hundredth, $.3\,x^2 - .02\,x = .5$ (N. Y. State Regents' Examination.)

273. At what times between 4 and 5 o'clock are the minute and hour hands of a clock equally distant from 4? (Teacher's Examination, Ohio.)

274. The numerator of a certain fraction exceeds its denominator by 5. If the numerator be increased by 3 and the denominator be decreased by 4, the sum of the new fraction and the original fraction is 3. Find the original fraction. (Cornell.)

275. Pure ground mustard contains 35% of oil. A sample of mustard is adulterated with wheat flour. The per cent of oil found in the sample is 15. Find the per cent of wheat flour in the mixture, allowing 2% of oil to exist naturally in wheat flour. (Teacher's Examination, Ohio.)

276. Using the same set of axes, represent graphically the following set of equations and from the graphs determine the solutions that the two equations have in common:

$$x + 3\,y = 12.$$

$$x^2 = 6\,y.$$ (Regents' Examination, New York.)

277. The sum of the squares of the two digits of a number is 58. If 36 be subtracted from the number, the digits of the remainder are the digits of the original number in reverse order. Find the number. (Cornell.)

278. Find to the nearest tenth the roots of the following: $2\,x^2 - 5\,x - 1 = 0$. (Teacher's Examination, New York.)

279. Find the value of k for which the graphs of the equations $x^2 + y^2 = 5$ and $y = 2\,x + k$ will have one and only one point of intersection. Draw the graphs of the two equations, using the value of k which you have found. (Cornell.)

280. (a) A man owns a house which he rents for r dollars a month. If the taxes and improvements for the year total t dollars and s dollars respectively, what is his yearly net income?

(b) Find the sum of a yards, $3c$ inches, and $2b$ feet $4c$ inches, and express the result in inches.

(c) If d stands for divisor, D for dividend, q for quotient, and r for remainder, write a formula for D expressed in terms of the other letters. (Teacher's Examination, New York.)

281. (a) Multiply $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 5$ by $x^{-\frac{2}{3}} + 2x^{-\frac{1}{3}} - 1$.

(b) Write the product with positive exponents in ascending powers of x.

(c) Write the answer in radical form. (Regent's Examination, New York.)

282. Find and simplify the 12th term of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{15}$. (Cornell.)

283. A's money added to $\frac{2}{3}$ of B's, which is to A's as 2 is to 3, being put on interest for 6 years at 4% amounts to \$744. How much money has each? (Teacher's Examination, Ohio.)

284. Simplify: $\dfrac{\dfrac{a}{a-b} - \dfrac{a}{a+b}}{\dfrac{b}{a-b} + \dfrac{a}{a-b}}$. (Cornell.)

285. Form the quadratic equation whose roots are $\frac{5 - \sqrt{-3}}{2}$ and $\frac{5 + \sqrt{-3}}{2}$, and write it in simplest form. (Cornell.)

286. An electrician and his helper together received \$11.55 for 7 hours' work. On another job the electrician worked 5 hours and his assistant $6\frac{1}{2}$ hours, and they received

$8.85. Find the wages of each per hour. (Teacher's Examination, New York.)

287. If 20 rods are added to the length of a certain rectangular field and 4 rods are subtracted from the width of this field, the area will be increased 40 square rods. If 10 rods are subtracted from the length and 5 rods are added to the width, the area will be diminished 50 square rods. Find the dimensions of the field. (Regent's Examination, New York.)

288. (a) Solve the formula $S = \pi(R + r)L$ for R in terms of S, r, π, and L.

(b) Find the value of R correct to the nearest tenth if $S = 527.33$, $r = 5$, $\pi = 3.14$, and $L = 9$. (Teacher's Examination, New York.)

289. Solve for x and check:

$$\frac{x-1}{x+1} - \frac{x+1}{1-x} = \frac{2x^2+3x-4}{x^2-1}.$$ (Cornell.)

290. (a) Without multiplying the expressions and extracting the root, find the square root of the following indicated product.

$$(6a^2 + 17ab + 5b^2)(9a^2 - b^2)(6a^2 + 13ab - 5b^2).$$

(b) Multiply $2\sqrt{6} - 4$ by $\sqrt{2} + 3\sqrt{3}$. (Teacher's Examination, New York.)

291. Two trains are scheduled to leave the cities A and B, m miles apart, at the same time, and to meet in h hours. But the train from A being a hours late in starting, and running at its regular rate, the trains meet k hours later than the scheduled time. What is the rate at which each train runs? (Cornell University, Scholarship Examination.)

292. If it is desired to dilute 2 quarts of a certain mixture containing a 5% solution of salt to one containing a $2\frac{1}{4}\%$ solution, how much water must be added to the mixture? (Teacher's Examination, New York.)

293. The legs of a right triangle are in the ratio 5 to 12. Its perimeter is 75 inches. Find the lengths of the three sides. (Teacher's Examination, New York.)

294. Due to a government tax, a customer is compelled to pay 44 cents extra for a certain amount of medicine. The druggist suggests that if the customer will take $\frac{1}{11}$ less than usual, the price, including the tax, will be the same as that formerly paid for the full amount. Find the rate of taxation. (Teacher's Examination, New York.)

295. A number of men agree to contribute equally toward purchasing a club house for $7500. Later five more apply for membership among the contributors, thus causing the individual assessment to be $50 lower. What was the amount of the original assessment per capita? (Teacher's Examination, New York.)

296. Three numbers are in geometrical progression. The sum of the first and second is equal to the fourth decreased by the first. If 2 were subtracted from the first, 2 from the second, and 8 from the third, the three numbers obtained would be in arithmetic progression. Find the numbers. (Cornell University.)

297. The reciprocal of a number is equal to four times the reciprocal of the sum of the number and 18. Find the number. (Teacher's Examination, New York.)

298. The length of a fence around a rectangular field is 274 yards and the distance measured diagonally from corner to corner is 97 yards. What is the area? (Teacher's Examination, New York.)

299. A detachment from an army was marching in column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men and the detach-

ment was thus drawn up in 5 lines. Find the number of men in the detachment. (Teacher's Examination, New York.)

300. Simplify:

$$\frac{(2^{-5})(3^4)\sqrt{\sqrt[3]{64}}}{(27)^{\frac{4}{3}}\sqrt{\frac{1}{16}}}.$$ (Cornell University.)

SHORT REVIEW ARRANGED BY CHAPTERS

1. Subtract $-x^2+y^2-z^2$ from the sum of $\frac{1}{3}x^2+\frac{1}{4}y^2$, $\frac{1}{5}y^2+\frac{1}{3}z^2$, and $\frac{1}{3}z^2-\frac{1}{4}x^2$.

2. Divide $x^2-xy+\frac{3}{16}y^2$ by $x-\frac{1}{4}y$, and check.

3. Find the value of

$$a(a+b)(a+b+c)-a(a-b)(a-b-c)$$

when $a=5$, $b=3$, and $c=-6$.

4. Show that

$$a^3+b^3+c^3-3abc-a(a^2-bc)-b(b^2-ca)-c(c^2-ab)=0.$$

5. Solve and check:

$$3x-3[4x-2(2x-5)]=9-2[3x-5(x-5)].$$

6. Solve and check:

$$.006x-.491+.723=-.005.$$

7. Factor:

(a) x^4-y^4.
(b) $x^2-3x-28$.
(c) $a^2-b^2+2a-2b$.
(d) x^3-4x^2+4x.
(e) $2x^2-5x+2$.
(f) x^3-x^2+x-1.
(g) $x^3+x^2y-6xy^2$.
(h) $x^3+ax^2-a^2x-a^3$.
(i) $x^2y^2-x^2-y^2+1$.
(j) $a^3-2abc-ab^2-ac^2$.

8. Simplify $\dfrac{1-x^2}{1+y}\cdot\dfrac{1-y^2}{x+x^2}\left(1+\dfrac{x}{1-x}\right)$.

9. Solve

$$5\{2x+1-3(x+1)\}-\frac{x+2}{3}+\frac{10}{7}(3x+5\tfrac{1}{2})+2\tfrac{3}{5}=0.$$

10. Solve $3 - \dfrac{12}{x+1} - 7 = -\dfrac{4x+30}{x+3}$.

11. Solve $\dfrac{a+x}{a+b} + \dfrac{ax}{b} = \dfrac{b}{a}$ for x, and verify by substituting numerical values for the different letters.

12. Solve for x and reduce the result to the simplest form:

$$\frac{1}{a^2} - \frac{1 + \dfrac{1}{a}}{\dfrac{a}{x}\left(1 - \dfrac{1}{a^2}\right)} = -\frac{1}{a}.$$

13. A man who can row four miles an hour in still water rows uniformly downstream for one hour. Then he floats with the current for half an hour, and rows back in three and one half hours. How rapid is the current?

14. A steamer can run 20 miles an hour in still water. If it can go 72 miles with the current in the same time that it could run 48 miles against the current, what is the speed of the current?

15. A man has \$1500 out at simple interest. For \$250 of it he receives 5%, for part of the remainder $3\frac{1}{2}$%, and for the rest 4%. If the total interest for 3 years is \$174, find how much is invested at $3\frac{1}{2}$%.

16. A formula for finding the proceeds of a bank note is $P = p(1 - rt)$, where

P = proceeds,
p = face of the note,
r = rate per cent,
t = time in years for which note is discounted.

Find the proceeds of a note for \$2500 discounted for 60 days at 6%.

17. Multiply

$5\sqrt{8} + 6\sqrt{12} - 2\sqrt{20}$ by $7\sqrt{2} - 3\sqrt{3} + 4\sqrt{5}$.

18. Replace $\dfrac{\sqrt{5}-3\sqrt{3}}{2\sqrt{5}-\sqrt{3}}$ by an equivalent fraction whose denominator is rational.

19. Reduce $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ to an equivalent fraction having a rational denominator by multiplying both numerator and denominator by $(\sqrt{2}+\sqrt{3})+\sqrt{5}$.

20. Find the value of $\sqrt{\dfrac{5}{x}}-\sqrt[3]{-x}$, when $x = .008$.

21. (a) Extract the square root of

$$4x^6+12x^5+5x^4-14x^3-11x^2+4x+4.$$

(b) Find the square root of the product of

$$x^2-1,\ x^2-3x+2, \text{ and } x^2-x-2.$$

22. Find the base and the altitude of a rectangle whose perimeter is 38 feet and whose area is 48 square feet. Solve your equation by factoring.

23. (a) Simplify $(a^{\frac{1}{n-1}}a^{\frac{1}{n+1}})^{\frac{1}{n}}$.

(b) Extract the square root of

$$1-\tfrac{3}{2}x^{\frac{1}{3}}+\tfrac{41}{16}x-\tfrac{3}{2}x^{\frac{3}{3}}+x^2.$$

24. Find the value of x from the equation

$$8^{\frac{1}{6}}+x^{-\frac{1}{3}}=\frac{7}{3-\sqrt{2}}.$$

25. (a) Which is larger, $2^{\frac{7}{3}}$ or 5? Prove your answer.

(b) Express as an integer or simple fraction the sum of

$$12^0+4\tfrac{1}{2}-9^{-1}+\frac{1}{\sqrt[3]{-64}}+27^{\frac{2}{3}}.$$

26. Find the value of x which will make $a^3-5a^2-11a-x$ divisible by a^2-7a+1.

27. Reduce each of the following:

(a) $(1 + \sqrt{x})^2$.

(b) $\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}} \cdot \frac{1}{(1-x)^2}$.

(c) $8^{-\frac{2}{3}} \div 256^{-\frac{1}{4}}$.

(d) $\frac{3\sqrt{1-x^2} - (1-x^2)^{-\frac{1}{2}}}{1-x^2}$.

28. Find the square root of

(a) $85 + 5\sqrt{120}$. (b) $38 - 12\sqrt{10}$.

29. Rationalize completely the denominator of

$$\frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}.$$

30. The number of gallons of water per minute flowing over a rectangular weir (an opening through which the water is allowed to pass) 6 inches wide is $g = 17 \cdot 8\, h^{\frac{3}{2}}$, where g = number of gallons per minute, h = depth in inches from the level of free water to the sill of the weir. Find the number of gallons per minute when $h = 4$ inches. If $h = 2$ inches.

31. Factor $x^4 - y^4$ into four factors.

Simplify:

32. $3\sqrt{-4\,a^4} + 4\sqrt{-9\,a^4} - 2\sqrt{-81\,a^4}$.

33. $\frac{11}{\sqrt{-11}}$. **34.** $\frac{3 + 2\sqrt{-3}}{3 - 5\sqrt{-3}}$.

35. Simplify each of the following:

$(-8\sqrt{-7})(-12\sqrt{-10})$; $12\sqrt{-4} \div 6\sqrt{-2}$; $(7 - 2i)^2$;

$\sqrt{-24} \div \sqrt{-6}$; $(\sqrt{-8})(-\sqrt{-18})(\sqrt{-32})$;

$\sqrt{-1} + \sqrt{-49} - \sqrt{-25}$; $\sqrt{14 + \sqrt{-60}}$; $\frac{4 + \sqrt{-3}}{4 - \sqrt{-3}}$.

Solve:

36. $12x^2 + 7x = 62$.

37. $\frac{12}{5-x} + \frac{8}{4-x} = \frac{32}{x+2}$.

38. $(x+2)(x-4) = 3(x+2)$.

39. $(2x+1)(x+2) = (x-1)^2$.

40. Find two consecutive even integers, the sum of whose squares is 244.

41. Telegraph poles are placed at equal intervals along a railroad. If there were to be two fewer poles per mile, it would be necessary to increase the space between every two consecutive poles by 24 feet. Find the number of poles to the mile.

42. A and B ran a half-mile race. A, who is faster than B by half a yard per second, allows B a handicap start of one fourth of a minute, and beats him by 5 yards. What are their respective rates in yards per second?

Determine, without solving, the nature of the roots of the following equations:

43. (a) $9x^2 - 13x + 4 = 0$. (c) $x^2 - 2x + 3 = 0$.
(b) $9x^2 - 24x + 16 = 0$. (d) $9x^2 - 12x + 2 = 0$.

44. Show that the roots of $x^2 - 2ax + a^2 = b^2 + c^2$ are real.

45. Solve $x^2 - 2mx + 3m^2 = 5$. State what values must be given to m to make the roots (a) real and equal, (b) imaginary and unequal.

46. Determine under what conditions the roots of the equation $ax^2 + bx + c = 0$ will be (1) both positive; (2) both negative; (3) one positive and the other negative.

47. Form the quadratic equation whose roots are (a) 8 and $\frac{1}{3}$; (b) $2 + \sqrt{3}$ and $2 - \sqrt{3}$; (c) $-\frac{2}{7}$ and 0; (d) $2 + \sqrt{-3}$ and $2 - \sqrt{-3}$.

48. Given that one root is $-\frac{5}{8}$, find, without solving, the remaining root of the equation $8x^2 - 3x - 5 = 0$.

Solve for x and y:

49. $3x + 4ay = 8ab.$

$$\frac{x}{7a} - 7y + 3b = 0.$$

50. $$\frac{y}{x-a} \cdot \frac{c}{b+a} = 1.$$

$$\frac{y}{x+a} = \frac{c}{b+a}.$$

What are the values of x and y thus found when $a = 2$, $b = 1$, and $c = 5$?

51. $$\frac{5}{x} + \frac{3}{y} = 1\tfrac{1}{2}.$$

$$\frac{3}{x} - \frac{2}{y} = \frac{4}{15}.$$

52. $x + y + 2z = 1.$
$2x + 3y + 2z = 4.$
$4x + 9y + 2z = 16.$

53. A train running 30 miles an hour requires 21 minutes longer to go a certain distance than does a train running 36 miles per hour. How great is the distance?

54. Two men are 25 miles apart, and walk toward each other at the rate of $3\frac{1}{2}$ and 4 miles an hour respectively. After how long do they meet?

55. Find all the values of x which satisfy the equation

$$\sqrt{x - 5a} - \sqrt{x + 3a} = \sqrt{x - 2a}.$$

56. Solve for x and y, grouping the corresponding values:

$$4x + 3y = 9$$
$$2x^2 + 5xy = 3.$$

57. Solve and group your results:

$$x(x - y) = 0.$$
$$x^2 + 2xy + y^2 = 9.$$

58. Find values of x and y that satisfy the equations

$$x^2 + 4y^2 = 4a^2 + b^2.$$
$$2x + y = 2b - a.$$

59. The square on the hypotenuse of a right triangle contains 180 square inches, the difference in the lengths of the legs of the triangle is 6 inches. Find the legs of the triangle.

60. The sum of the circumferences of two circles is 44 inches, and the sum of their areas $78\frac{4}{7}$ square inches. Find the radii of the two circles if $\pi = \frac{22}{7}$.

61. The perimeter of a rectangular field is 274 yards, and the diagonal from one corner to the opposite corner is 97 yards. Find the dimensions. Find the area.

Construct graphs of portions of the following equations; determine the roots when possible:

62. $y = -3x + 5.$

63. $y = x^2 + x.$

64. $y = -2x^2 + 3x - 5.$

65. $.7x = .07.$

66. $y = x^3 + x^2 + x + 1.$

67. $y = kx^2.$ Assume $k = 1.$

68. Graph the equation $x + 3y = 6.$

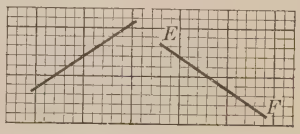
Multiply both members of this equation by 2 and graph the new equation. Also multiply by 3 and graph the resulting equation. Compare your results.

69. Graph the equation $x + 3y = 12.$ Also the equation $x + 3y = 18.$ Compare your graphs with the first graph of example 68. What conclusions can you draw from these graphs?

70. The *slope of a line* is the ratio of the change of the value of y to the corresponding change of the value of x as we think of the point moving along a line or of the different points that we may think of on the line. The general slope equation is $y = mx + b$.

m means the ratio explained above.

b is the distance from the origin to the point where the line cuts the Y-axis. For example, in the equation $2x - 3y + 6 = 0$, we can solve this for the value of y and we obtain $y = \frac{2}{3}x + 2$. Now forming a small table of values and plotting the equation as shown in the figure, we find that $b = 2$, and that as x increases 3, y increases 2. Then $m = \frac{2}{3}$. Therefore we say that the slope of this line is $\frac{2}{3}$, which means that the line rises two units in a horizontal distance of 3 units. The slope of the line EF in the figure is $-\frac{2}{3}$, since the line falls or drops 2 in a horizontal distance of 3.



In a similar manner construct the graph of the line or of the equation

$$3y = 2x + 9.$$

What is the slope of this line? What is the value of b?

71. Construct the locus of the equations:

(a) $4y - 3x = 4$.

(b) $4y - 3x = -4$.

(c) $y = \frac{2}{5}x$.

Hint: What is the value of b?

(d) $2y + x = 4$.

72. The *demand curve* may be used to prophesy what will happen at some future time under certain given conditions. The population of New York and Philadelphia are given below by ten-year periods.

Population in Thousands (Round Numbers)

	New York	Philadelphia
1820	152	64
1830	242	80
1840	391	94
1850	696	121
1860	1175	566
1870	1478	674
1880	1912	847
1890	2507	1047
1900	3437	1294
1910	4767	1509
1920	6002	1823

Construct two curves on the same chart field to represent the above facts. Assume that these cities have a normal growth, continue your curves to show the demand upon the transit and telephone facilities of these cities in 1930. In 1950. In 1980. In 2000.

73. If a fruit dealer could sell 50 baskets a day when his price was \$1.50 per basket; 125 baskets at \$1.00; 250 baskets at \$.75; 500 baskets at \$.50, construct a curve graph to show these facts. From your graph estimate the demand at \$1.25 a basket. At \$.90 a basket. At \$.40 a basket.

74. What is the value of the ratio $\frac{x}{y}$ in $(a + b)x = (a - b)y$ when $a = 9$ and $b = 3$?

75. The areas of two similar plane figures have the same ratio as the squares of any two corresponding dimensions. If the area of a given triangle is 80 square inches and the base is 16 inches, what is the area of a similar triangle whose base is 20 inches?

76. The area of a circle varies as the square of the diameter. If the area of a circle, the radius of which is 21 feet, is 1386 square feet, what is the area of a circle whose radius is 16 feet?

77. The 9th term of an arithmetic progression is $\frac{1}{6}$; the 16th term is $\frac{5}{6}$. Eliminate l between your two equations thus formed and from your results find the first term.

78. If the cost of boring a well is 27 cents for the first foot, 28 cents for the second foot, and so on, how much will it cost to bore a well 400 feet deep?

79. The sum of three numbers in geometrical progression is 14 and their product is 64. Find the numbers.

80. B travels 3 miles the first day, 7 miles the second day, 11 miles the third day, etc. In how many days will B overtake A, who started from the same point 8 days in advance and who travels uniformly 15 miles a day?

81. A rubber ball falls from a height of 40 inches and on each rebound rises 40% of the previous height. Find, by formula, how far it falls on its eighth descent.

82. Expand by the binomial theorem $\left(\frac{x^2}{3} - 2a^{-1}\right)^5$.

83. Find and simplify the ninth term of $\left(a^2 - \frac{2}{a^{\frac{1}{2}}}\right)^{12}$.

84. Compute the value of $\frac{(57.1)^2 \cdot .9004}{721}$ by logarithms.

85. Find the value of x which reduces the following determinant to 0: $\begin{vmatrix} 5 & 1 & x \\ 8 & x & 2 \\ 4 & x & 4 \end{vmatrix}$

In the following examples, use the table, page 455, following the material of the text.

Calculate the value, when not integral, of each of the following to the nearest hundredth:

86. The cube root of 328,509. The square root of 8281.

87. (a) $12\sqrt[3]{28}$. (b) $\frac{3}{4}\sqrt{38}$. (c) $\dfrac{20\sqrt{11} - 5\sqrt[3]{60}}{3}$

88. The area A of the cross section of a chimney in square feet, which is necessary to carry off the smoke, is indicated by the formula $A = \dfrac{.6\,P}{\sqrt{h}}$

where P = number of pounds of coal burned per hour,
h = height of chimney in feet.

What is the area of the cross section of a chimney 70 feet high in order to carry off the smoke if 500 pounds of coal are burned per hour?

Also work this problem by first rationalizing the denominator and compare the amount of work by the two methods.

89. In the equation $x = 10^y$ what is the value of x when $y = 0$? When $y = 2$? When $y = \frac{1}{2}$? When $y = \frac{1}{3}$? When $y = -2$?

90. Find the value of the expression

$$x^{2n} + 2\,x^n y^m + y^{6m},$$

if $x = 15$, $n = \frac{1}{2}$, $y = 56$, $m = \frac{1}{3}$.

91. Find the square root of 4862 to tenths.

Solution: 4862 is not in the table. Find the number of the square column just below and the one just above 4862. Arrange as follows:

4900 Subtract the lower number from the middle one,
4862 then the lower from the higher one. Form a fraction with these differences as follows: $\frac{101}{139} = .7 +$.
4761

The square root of 4761 is 69. We have then for our answer 69.7 +. This method is generally approximately

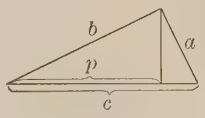
correct to tenths. In a similar manner find the square root of (a) 7856. (b) 3425.

92. By a similar plan to that explained in example 91, find the cube root of (a) 9864. (b) 580,000.

93. By first simplifying the radicand and then using the table find the value of

$$\tfrac{5}{8}\sqrt{250} + \tfrac{1}{2}\sqrt{72} + \tfrac{1}{4}\sqrt[3]{40}.$$

94. In the adjoining triangle the side $a^2 = b^2 + c^2 - 2cp$. Find a, if $b = 13$, $c = 14$, $p = 12$.



RECREATIONS AND FALLACIES

1. State a word of five letters of which if you take two away you will then have one.

2. How old is Dad?

There's Dad and Ma, Brother and Me.
The sum of our ages is 83.
6 times Dad's age is 7 times Ma's.
And she is three times Me.

3. The Man and the Stores.

A man goes to a store, and says, "Give me as much as I have, and I will spend ten cents with you." It is done, and the man repeats the operation at a second and a third store, and has no money remaining. What did he start with?

4. The Wine Casks.

Two men find in a cellar an eight-gallon cask full of wine, also an empty five-gallon and an empty three-gallon cask. They wish to measure out two lots of four gallons each. How can it be accomplished?

5. Let a and b be two unequal numbers, and let c be their arithmetical mean. Then

$$a + b = 2c.$$
$$\therefore (a + b)(a - b) = 2c(a - b).$$
$$\therefore a^2 - 2ac = b^2 - 2bc.$$
$$\therefore a^2 - 2ac + c^2 = b^2 - 2bc + c^2.$$
$$\therefore (a - c)^2 = (b - c)^2.$$
$$\therefore a - c = b - c.$$
$$\therefore a = b.$$

Note with what we started, and find the fallacy in this reasoning.

6. Draw four straight lines so as to form four triangles.

7. Mile Square and Square Mile.

A mile square is equal to a square mile. Is $\frac{1}{2}$ mile square equal to $\frac{1}{2}$ of a square mile?

8. Prove that two thirds of 6 is 9, one half of 12 is 7, one half of 5 is 4, and 6 is one half of 11.

9. Suppose:

If the fourth of 20 were 3,
What would the third of 10 be?

10. We can write $\sqrt{-1} = \sqrt{-1}$ in the form of

$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}}.$$
$$\therefore (\sqrt{-1})^2 = (\sqrt{1})^2.$$
$$\therefore -1 = 1.$$

Show the fallacy.

11. How many of each?

"Good morning, sir, with your hundred geese and ducks." "No, sir, I have not a hundred; but if I had twice as many geese or three times as many ducks, I would have a hundred in all."

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

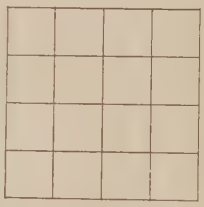
12. We have $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

Hence $\sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)}$.

$\therefore$ $(\sqrt{-1})^2 = \sqrt{1}$.

That is, $-1 = 1$.

State the fallacy.



13. Arrange the first 16 numbers (1–16) in the squares, no number to be used but once, so as to add up to 34 in any row, column, or diagonal.

14. Arrange the first nine numbers in a similar square having three small squares each way.

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$\sqrt{3} = 1.732$

$\sqrt{2} = 1.414$

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